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THE GENERATION OF RANDOM NUMBERS
FROM VARIOUS PROBABILITY DISTRIBUTIONS

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THE GENERATION OF RANDOM NUMBERS
FROM VARIOUS PROBABILITY DISTRIBUTIONS

* * * * *

John E. Howe

THE GENERATION OF RANDOM NUMBERS
FROM VARIOUS PROBABILITY DISTRIBUTIONS

by

John E. Howe

Lieutenant, United States Navy

Submitted in partial fulfillment of
the requirements for the degree of

MASTER OF SCIENCE
IN
OPERATIONS RESEARCH

United States Naval Postgraduate School
Monterey, California

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FROM VARIOUS PROBABILITY DISTRIBUTIONS

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This work is accepted as fulfilling
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IN

OPERATIONS RESEARCH

from the

United States Naval Postgraduate School

ABSTRACT

Methods are developed, and Fortran 63 CODAP computer programs are demonstrated, to generate random numbers from the uniform, normal (including multivariate normal), Poisson, and exponential probability distributions. Various statistical tests are described and the results of the application of these tests to the generators are tabulated. A general method for generating random numbers from a large class of distributions is described. The methods of generation are optimized to provide an accurate generator while producing numbers at a maximum rate. The uniform generator that is used as a basis for the other generators is of the congruential type and is capable of generating 1800 numbers per second.

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13	Apr 30 Cash	5.00
14	May 10 Cash	25.00
15	May 20 Cash	15.00
16	May 30 Cash	10.00
17	Jun 10 Cash	35.00
18	Jun 20 Cash	25.00
19	Jun 30 Cash	15.00
20	Jul 10 Cash	45.00
21	Jul 20 Cash	35.00
22	Jul 30 Cash	25.00
23	Aug 10 Cash	55.00
24	Aug 20 Cash	45.00
25	Aug 30 Cash	35.00
26	Sep 10 Cash	65.00
27	Sep 20 Cash	55.00
28	Sep 30 Cash	45.00
29	Oct 10 Cash	75.00
30	Oct 20 Cash	65.00
31	Oct 30 Cash	55.00
32	Nov 10 Cash	85.00
33	Nov 20 Cash	75.00
34	Nov 30 Cash	65.00
35	Dec 10 Cash	95.00
36	Dec 20 Cash	85.00
37	Dec 30 Cash	75.00
38	Total	1000.00

1. Introduction.

The need for a rapid reliable source of random numbers from a prescribed distribution is well recognized, and is likely to become even more pressing in military circles in view of the Department of Defense's increased interest in Operations Analysis. This thesis presents methods and programs, some old, some new, for generating random numbers from various specified distributions. Some statistical tests of the programs and their results are described.

All methods assume a uniform random number generator is available. A thesis by Barron [1] has a good bibliography on this subject. The generator used here, and described more completely in Section 3, is of the mixed congruential type. While some uniform generators may have advantages over the one used, this one seems to perform very well, at the same time as being as fast as any demonstrated. Since this generator is used as a basis for all the others it should be remembered that no generator can be considered 'perfect', especially in the continuous distribution case, since the computer is limited to a finite set of possible numbers. However, for practical purposes this inaccuracy is not important. Other sources of inaccuracy can be important, however. The numbers generated must have the property of randomness, and must faithfully represent the desired distribution. These properties are measured by testing samples of the generated numbers. Some of the methods of generating numbers in this paper theoretically provide an exact

transformation from the uniform distribution to the desired distribution. An example of this is the half-Gaussian method of generating normal random numbers¹. If we assume that the uniform numbers are accurate then we are led to the conclusion that the normal numbers are also accurate and that tests of these numbers would be superfluous. However, tests are performed to assure that the uniform numbers were so good that they did not bias the derived distribution. Other techniques such as Marsaglia's technique (see Section 4) for generating normal random numbers are in a sense curve-fitting techniques and only provide a controllably good approximation to the real distribution. The advantage of the approximation techniques is in the far greater speed with which they may provide the desired numbers. The user who needs to draw numbers from a distribution he suspects is normal, with inaccurately measured mean and hypothesized variance, is not in need of a generator that is accurate in the sixth decimal place, however, he still would like to be assured that having assumed a distribution and its parameters he will be able to generate numbers with the appropriate shape and with good properties of randomness.

¹The phrase 'normal random numbers' and other like phrases should be read as 'numbers distributed as if coming from the normal distribution.'

2. Testing the Generators.

2.1 General Discussion.

The uniform generator chosen here has been tested extensively by others. The tests that have been applied include frequency tests, serial tests, moment tests, poker tests, gap tests, and many others. As mentioned in the introduction, this generator is as good as can be found, considering the requirement for speed. However, the derived distributions will be tested to overcome any doubts there may be about the transformation. The numbers will be tested mainly to measure the faithfulness with which they represent the derived distribution. The randomness is provided by the uniform generator. If the randomness is not satisfactory then the uniform generator must be blamed, not the transformation. A slower but more satisfactory method is available if the need is felt.¹ Thus the tests used here, the moment test, the hypothesis tests on the mean and variance, and the Kolmogorov-Smirnov goodness of fit test, are not designed to detect special types of non-randomness, such as is detected by the poker test and other similar tests.

Martin Greenberger has written an interesting article [17] on this subject. He presents the results of an investigation by Joseph Lach [18] at Yale University in which Lach showed that the congruential method of generating uniform random numbers has a

¹See page 22.

predictable non-randomness in second order serial correlation. The lesson is clear. Having found an undesirable feature in a generator, it is generally possible to modify the generator to eliminate the feature, however, we can be sure that we have introduced another aberration of some kind, even though its form may be hard to determine. When the user asks, "Is this generator good enough?", the obvious retort is "good enough for what?" No one generator is suitable for all applications, but the generator used here will be good enough for most. If the user thinks that this is not so in his application, he has at his resources the modifying methods of Marsaglia [8] or Lach with the penalty of longer generation times.

2.2 Moment tests

The first four moments are calculated and are compared with the theoretical moments for each of the distributions. A more appealing statistic than the sample moment may be the unbiased estimator of the moment, however for large sample sizes such as are used here, this varies very little from the sample moment, and the sample moment is easier to handle in other uses- such as hypothesis testing. The unbiased estimator of the variance is:

$$\hat{\sigma}^2 = \frac{1}{N-1} \sum (x_i - \bar{x})^2$$

The statistic used here is the sample second moment:

$$M2 = \frac{1}{N} \sum (x_i - \bar{x})^2$$

¹All numerations are on the index 'i' which runs from 1 to N, the sample size, unless otherwise indicated.

The first moment is the mean:

$$\bar{X} = M1 = \frac{1}{N} \sum X_i$$

The third and fourth moments are:

$$M3 = \frac{1}{N} \left[\left(\sum X_i^3 \right) - \frac{3}{N} \left(\sum X_i \right) \left(\sum X_i^2 \right) + \frac{2}{N} \left(\sum X_i \right)^3 \right]$$

$$M4 = \frac{1}{N} \left[\left(\sum X_i^4 \right) - \frac{4}{N} \left(\sum X_i \right) \left(\sum X_i^3 \right) + \frac{6}{N^2} \left(\sum X_i \right)^2 \left(\sum X_i^2 \right) - \frac{3}{N^3} \left(\sum X_i \right)^4 \right]$$

2.3 Hypothesis tests on the mean and variance.

The availability of large sample sizes is used in designing tests on the mean and variance. The central limit theorem is used where possible to simplify the test procedures.

2.3.1 Test on the mean.

This test is applied to the normal distribution. The hypothesis is that the mean is zero; the alternate hypothesis is that the mean is not zero. The test is performed by calculating the statistic Y, where

$$Y = \sqrt{N} \bar{X}$$

The decision rule at the alpha level becomes:

$$\text{Accept the hypothesis when } -K_{\alpha/2} < Y < K_{\alpha/2}$$

At the 10% level this rule becomes:

$$\text{Accept the hypothesis when } -1.645 < Y < 1.645$$

The justifications for using this test in preference to the 'T' test are that the variance can be assumed to be one, and that a large sample size is being used.

2.3.2 Test on the variance.

Again for the normal distribution, the hypothesis is that

the variance is one; the alternate hypothesis is that the variance is not one. The test is performed by calculating the statistic Z, where

$$Z = \frac{S - (N-1)}{\sqrt{2(N-1)}}$$

$$S = \sum (x_i - \bar{x})^2$$

The decision rule at the 10% level becomes:

Accept the hypothesis when $-1.645 < Z < 1.645$.

2.4 The Kolmogorov-Smirnov goodness of fit test.

In Massey's discussion of this test [6], he presents evidence to indicate that this test may in many circumstances be better than the more usual chi-squared goodness of fit test. To test the hypothesis that a sample of size N comes from theorized distribution, the cumulative step function $S_N(x)$ is formed.

$$S_N(x) = k/n$$

where k is the number of observations less than or equal to x. The selection of x is arbitrary within certain limits. In this paper x is chosen so that there are either twenty or fifty equal intervals spanning the sample space. $S_N(x)$ is compared with the theoretical value of the cumulative distribution, $F(x)$. The maximum difference d is calculated.

$$d = \max |F(x) - S_N(x)|$$

Tables due to Smirnov [7] give certain critical points of the distribution for various sample sizes. For sample sizes over 35, and at the 10% level of significance, if

$$d/n < 1.22/\sqrt{N}$$

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the sampled distribution is accepted as the hypothesized distribution.

2.5 The chi-squared goodness of fit test.

This test was developed by Pearson and represents the earliest non-parametric decision-making test in statistics. Partly because of the length of time the test has been in existence, many drawbacks to the test have been noted, however it still stands as a useful and much used test. For this test the sample space has been divided into k intervals and the number of sample observations in each interval is noted.

$$\chi^2 = \sum_{i=1}^k \frac{(q_i - m_i)^2}{m_i}$$

where q_i is the observed number of sample observations in each interval, and m_i is the expected number. Pearson showed that χ^2 has the chi-squared distribution with $k-1$ degrees of freedom.

The decision rule at the alpha level becomes: if $\chi^2 \leq \chi_{k-1}^2(\alpha)$ accept the hypothesis that the distribution is as postulated. There are several problems associated with the application of this test.

How should the interval size be determined? How many intervals should there be? Mann and Wald [14] studied this problem and formulated a criterion for the selection of k . Williams [15] notes that this criterion is not particularly sensitive to even a reduction by a factor of two in the number of intervals. The use of equal probability intervals vice equal length intervals is also recommended. However, the basis for this recommendation is unclear. It is agreed that very low probability intervals such as would occur

in the tails in an equal interval length division of the normal density function should be avoided. The test is applied here using equal length intervals. Low probability intervals are avoided by 'pooling' several intervals until the probability is of the same order of magnitude as in the other intervals.

2.6 Scatter diagrams

The best type of test to apply to the generator initially is some type of scatter diagram. The scatter diagram can often immediately give an intuitive idea as to whether the generator is behaving properly. In fact the scatter diagram can be a very powerful tool for rejecting a generator- more sophisticated techniques are needed to accept the generator, however. The scatter diagram that was used here was constructed by plotting the first number generated versus the second, the third versus the fourth, and so on. This type of plot will also enable us to look for correlations similar to those found by Lach [18] and discussed further in Section 2.1.

3. The Uniform Distribution.

3.1 Distribution characteristics.

It is desired to generate numbers such that:

$$f_x(x) = \begin{cases} 0 & x \leq 0 \\ 1 & 0 < x < 1 \\ 0 & x \geq 1 \end{cases}$$

The first four moments are: $M_1 = 1/2$; $M_2 = 1/12$; $M_3 = 0$;

$M_4 = 1/80$. The uniform distribution is usually specified in terms of its interval - uniform on the interval from zero to one (denoted here by $U(0,1)$). In the general case $U(A,B)$, a simple transformation from $U(0,1)$ is:

$$URAB = (UR01)(B-A) + A$$

where $UR01$ is the number provided by the $U(0,1)$ generator, and $URAB$ is the random number uniform on the interval (A,B) .

3.2 Methods of generation.

3.2.1 Many techniques have been used over the years. For some particular applications such a method as table look-up may be suitable. However for our purposes what is desired is a rapid, 'accurate' method for the computer to produce a practically inexhaustible supply of numbers.

3.2.2 An early computational scheme was called the mid-square method. In this procedure two starting values, say A_1 and A_2 , are multiplied together; the middle set of bits (usually 24) are extracted as the third random number A_3 ; then A_2 and A_3 are multiplied together and the algorithm is continued in a similar fashion. This method has performed well in many tests but

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unfortunately degenerates to all zeros in a relatively short time.

3.2.3 An improved computational scheme was tested extensively by Hull and Dobell [3] . This method forms a series of numbers A_i , where

$$A_{R+1} = BA_R + C \quad \text{modulus } 2^{47}$$

B and C are constants to be selected.

3.2.4 A special form of this generator where $C = 0$ is the generally recommended form. C is set equal to zero because it does not improve the characteristics of the generator and it adds to computation time. The selection of B and the starting number X has been the subject of considerable study. Barron and others have shown that:

$$X_{R+1} = (2^{19} + 3)X_R \quad \text{modulus } 2^{47}$$

where X is either 1, or $2^{48} - 1$, or any number naturally generated in the sequence, is an excellent generator. Since this generator had been tested extensively previously no attempt was made to test it rigorously although several interesting characteristics were noted.

Some of these are included here. The generator was run down through the first 10^7 to 10^8 numbers. A number at the end of this sequence was extracted for possible alternate use as a starting number. It can be found in Appendix I. A graph of the mean of the first 10000 times i numbers, where i runs from 1 to 100 is plotted against the index i. This plot is compared with curves of $= K_{.05} / 10000i + 0.50000$ versus i. The more our data plots between these curves the better.

We would expect it to be between the curves 90% of the time.

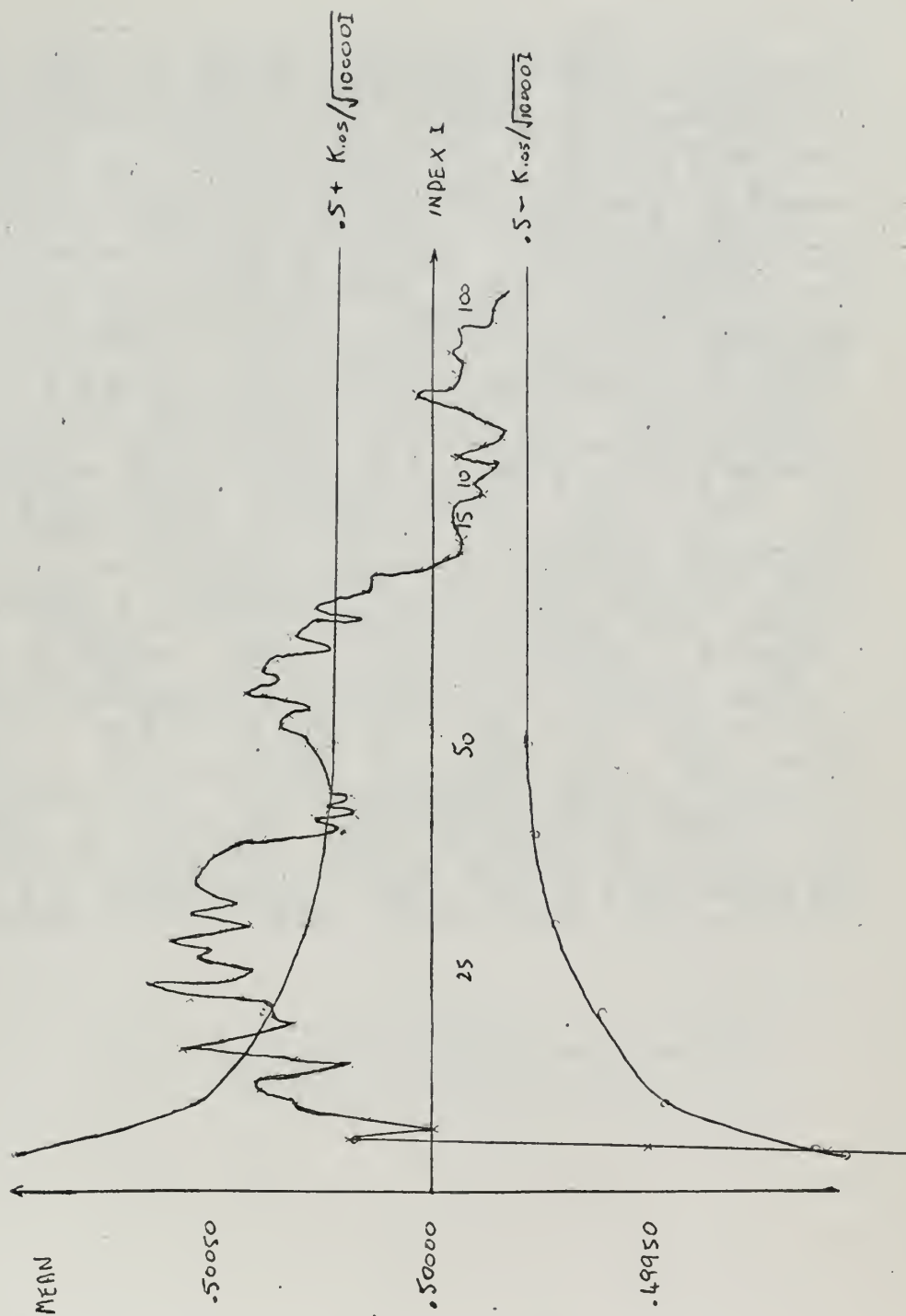
As the following graph shows our generator does not quite live up to this expectation. A scatter diagram of the type suggested by Lach [18] was plotted. The graph on page 13 consists of 3000 points constructed as described in Section 2.6.

3.2.5 George Marsaglia and M. Donald MacLaren [8] at Boeing Scientific Research Laboratories suggest that the combination of two generators will produce a superior random number of generator. They have tested:

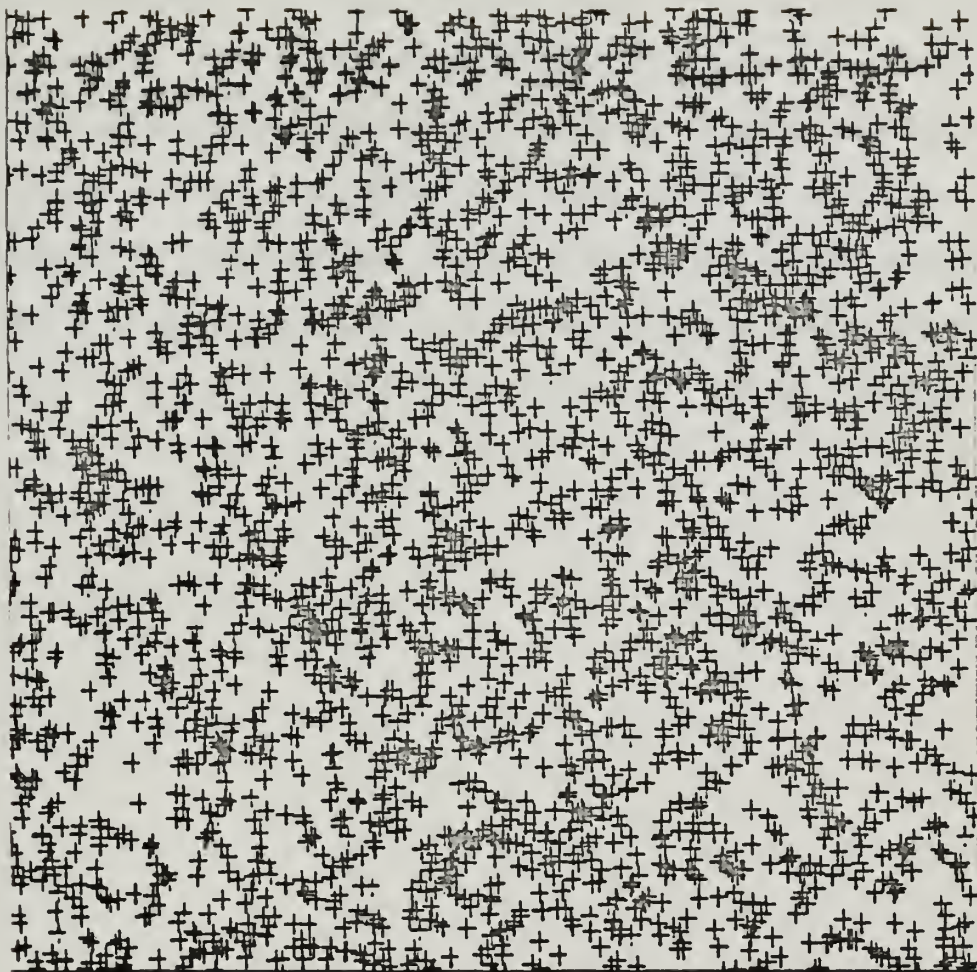
$$A_{R+1} = (2^{17} + 3)A_R \text{ modulus } 2^{35}$$
$$\text{and } B_{R+1} = (2^7 + 1)B_R \text{ modulus } 2^{35}$$

In essence they have used one generator to select numbers from the other. This generator seems to provide an improvement in some local randomness properties. However, the penalty for the improvement is doubling the time of generation. Marsaglia and MacLaren also noted that the method of table look-up may once more become feasible. In the case of the CDC 1604 this method is not practical. However, in a parallel program computer a method using a pair of generators may well be advantageous. The generators continually fill up the bottom of a short table in memory as the main program uses numbers from the top. The size of the table is chosen to ensure that the program never uses all the numbers in the table and thus the effective generation time will be just the load cycle time.

3.2.6 The CODAP, Fortran 63, machine language program for the



Cumulative Mean Of the First I Samples
Of The Uniform Random Number Generator
(with 1000 numbers per sample)



Scatter Diagram
For The Uniform (0,1) Random Number Generator
(Scale: 0.2 units per inch)

uniform generator used as a basis for this thesis is in Appendix I. The expected time of generation per number, as calculated over several samples of varying size was found to be 552 microseconds per number. This amounts to producing 1811 numbers per second. However, the generator is theoretically much faster than that. The time per number as calculated from times in the Control Data Corporation specifications for the computer is 121 microseconds. Measured times, depending on the context and the timing mechanism varied from a minimum of 370 to a maximum of 700 microseconds.

4. The Normal Distribution (Univariate Case).

4.1 Distribution characteristics.

It is desired to generate numbers such that the density function will be

$$f(y) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2}\left(\frac{y-u}{\sigma}\right)^2\right)$$

where u is the mean of the distribution and σ^2 is the variance. For the basic case the mean is taken to be zero and the variance to be one. The first four moments are $M_1 = 0$, $M_2 = 1$, $M_3 = 0$, $M_4 = 3$. If the desired distribution is to have a mean other than zero, say u , and a variance other than one, say V , then the following transformation applied to the numbers generated by the $N(0,1)$ generator developed here, represented by $RNO1$, will produce a number, $RNUV$, with the desired characteristics.

$$RNUV = (RNO1) (V) + U$$

In this paper the normal distribution is treated in three separate sections. The univariate case is developed first, then the bivariate, and finally a general multivariate case is demonstrated. The main purpose of this separate treatment is to allow a more efficient handling of the more commonly used univariate and bivariate cases. A general n -dimensional normal random number generator would be much slower, when used for n equals one, than the univariate generator demonstrated in Section 4. The test procedures for each generator are also different.

4.2 Methods of generation.

4.2.1 The normal distribution is one of the most used and

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tabulated distributions. As with the uniform distribution several procedures have been used to produce random numbers from it. The methods discussed here are those that are most adaptable to use on a computer.

4.2.2 The most common procedure has been to take the sum of K uniform random numbers. The central limit theorem shows that this (with the mean subtracted, and divided by the standard deviation) approaches the normal as K gets large. Vaa tested a generator using the sum of twelve uniform random numbers. This approximation has the disadvantage of being truncated at plus and minus six. Even more important a factor is the time required to generate these numbers. It is hoped that a more exact and faster method can be found.

4.2.2 The so-called half-Gaussian method [11] provides a theoretically exact transformation from the uniform to the normal. However in an attempt to reduce time some fairly drastic approximations have been made. These approximations should not affect randomness, however, and should only be a factor in accuracy beyond the fifth decimal place. The following flow chart is the basis for the routine. $R(J)$ is a uniform random number. HGRN is the normal number.

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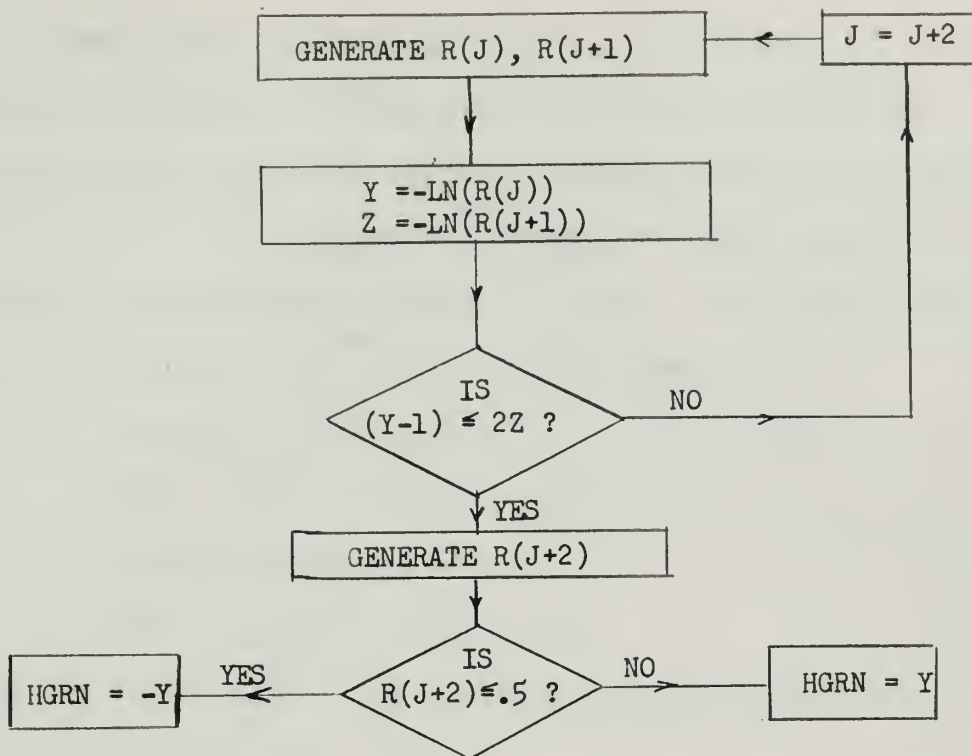
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The routine first generates the positive half of a normal distribution then adds a random sign selected by another uniform number. This program makes no external calls to the log function but rather uses the following series approximation:

$$T = (R-1)/(R+1)$$

$$\text{LN}(R) = 2(T + T^3/3 + T^5/5 + T^7/7 + T^9/9)$$

The CODAP function sub-program is Appendix II.

4.2.3 An excellent approximation technique has been developed by Marsaglia and Bray [9]. Marsaglia has developed several similar techniques but the one proposed here seems optimum in terms of time required per number and the storage space required. The method involves selecting one of four functions of varying complexities to produce the random number. One function is very simple and fast and

is used 86% of the time; the next function is also fast and is used 11% of the time. The remaining three percent of the time much more complex functions are used, however, due to the rapidity with which 97% of the numbers are formed, the overall expected generation time per number is relatively low. The program outline is as follows (MSRN is the desired random number):

1. 86.38% of the time, set

$$\text{MSRN} = 2[R(J) + R(J+1) + R(J+2) - 1.5]$$

2. 11.07% of the time, set

$$\text{MSRN} = 1.5[R(J) + R(J+1) - 1]$$

3. 2.28002039% of the time form pairs (X,Y) such that

$$X = 6R(J) - 3 \quad \text{and}$$

$$Y = 0.358 R(J+1)$$

until $Y \leq G_3(X)$; then set $\text{MSRN} = X$. $G_3(X)$ is defined by:

$$17.49731196 \exp(-x^2/2) - 4.73570326(3-x^2) - 2.15787544(1.5-|x|) \quad \text{for } |x| < 1$$

$$17.49731196 \exp(-x^2/2) - 2.36785163(3-|x|)^2 - 2.15787544(1.5-|x|) \quad \text{for } 1 < x < 1.5$$

$$17.49731196 \exp(-x^2/2) - 2.36785163(3-|x|) \quad \text{for } 1.5 < x < 3$$

4. 0.26997961% of the time form pairs (X,Y) until either X

or Y is greater than three, then let that one equal MSRN. For

$\text{RM}(J)$ uniform on the interval $(-1,1)$ and such that if $\text{RM}(J)^2 + \text{RM}(J+1)^2 \leq 1$,

then let $Z = \text{RM}(J)^2 + \text{RM}(J+1)^2$

and

$$X = \text{RM}(J) \left[\{9 - 2\text{LN}(Z)\} / (Z) \right]$$

$$Y = \text{RM}(J+1) \left[\{9 - 2\text{LN}(Z)\} / (Z) \right]$$

The CODAP function sub-program is Appendix III.

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4.3 Selection of generators.

A complete table of results of the tests on the normal generators is in Appendix IV. Partial results are presented below.

Average observed generation time per number	HALF-GASSIAN TECHNIQUE			MARSAGLIA TECHNIQUE		
	3625 microseconds			1503 microseconds		
Sample range	1-1000	1-10000	10001-20000	1-1000	1-10000	10001-20000
Sample size	1000	10000	10000	10000	10000	10000
Mean (theor. = 0)	0.04	-0.01	0.00	0.00	-0.02	-0.01
Variance (theor. = 1)	1.06	1.04	1.01	.99	1.00	.99
3rd Mom. (theor. = 0)	-0.05	-0.02	0.01	.01	-0.04	-0.07
4th Mom. (theor. = 3)	3.17	3.20	3.12	2.77	2.97	2.93
$\sqrt{N} \bar{X}$ (K _{.05} =1.64)	1.32	-0.96	0.30	0.13	-2.47	-1.06
$\frac{S-(N-1)}{\sqrt{2(N-1)}}$	1.30	2.57	0.95	-0.14	-0.11	-0.80
D_{\max}/N	0.0270	0.0073	0.0033	0.0082	0.0095	0.0049
$1.22/\sqrt{N}$	0.0386	0.0122	0.0122	0.0386	0.0122	0.0122

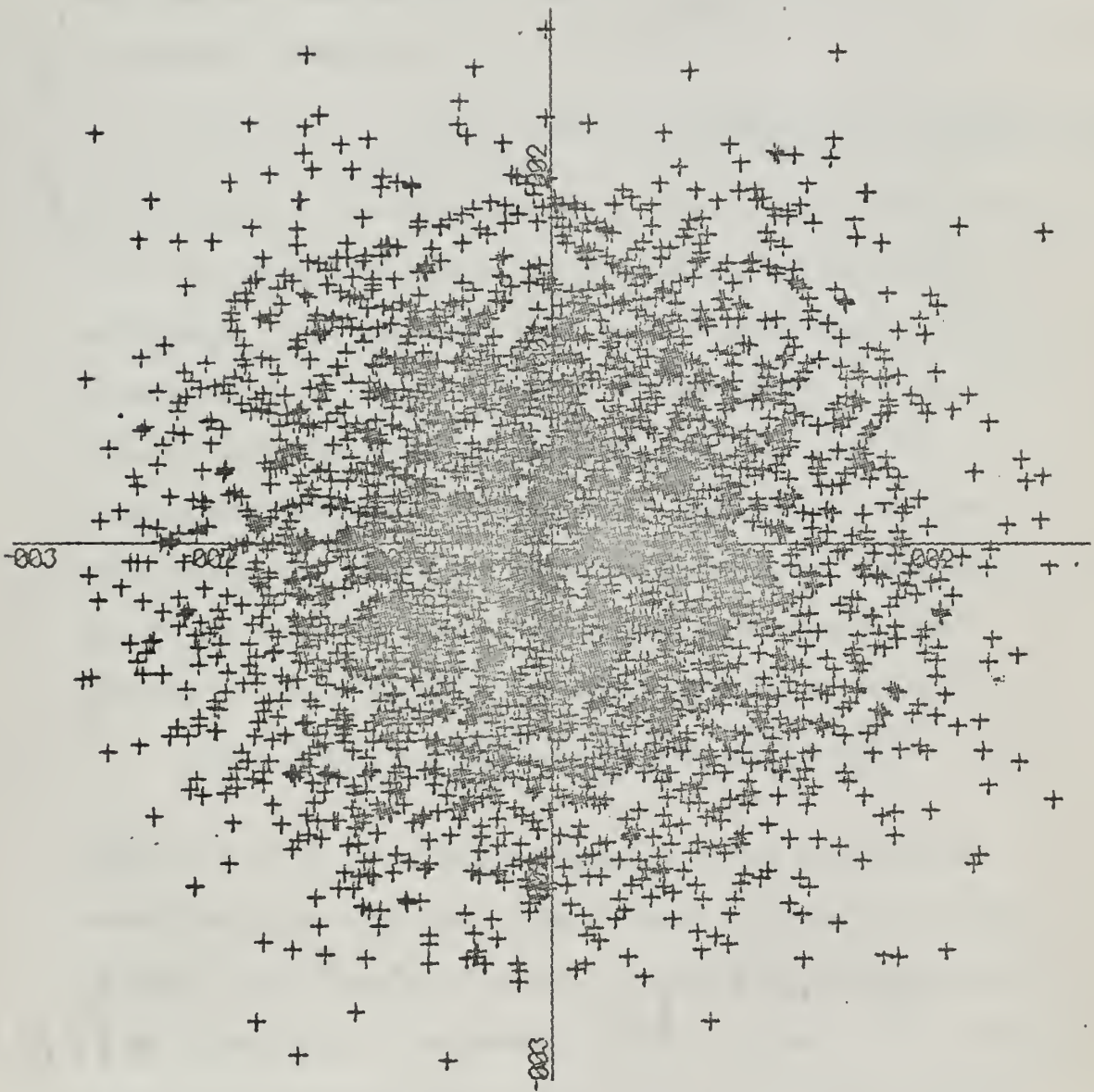
The samples generated by the half-Gaussian technique passed the hypothesis on the mean, at the 10% level, 80% of the time; on the variance at the 10% level, 30% of the time. At the 5% level the test on the mean was passed 90% of the time, the test on the variance was passed

1. *General description of the object*
 2. *Location of the object in the sky*
 3. *Physical properties of the object*

General description			Location in the sky			Physical properties
Name	RA	Dec	Galactic	Equatorial	Galactic	
1. <i>Object name</i>						
2. <i>RA (J2000)</i>						
3. <i>Dec (J2000)</i>						
4. <i>Galactic longitude</i>						
5. <i>Galactic latitude</i>						
6. <i>Equatorial longitude</i>						
7. <i>Equatorial latitude</i>						
8. <i>Distance (pc)</i>						
9. <i>Mass (M_{\odot})</i>						
10. <i>Radius (R_{\odot})</i>						
11. <i>Temperature (K)</i>						
12. <i>Age (Myr)</i>						
13. <i>Composition</i>						
14. <i>Other properties</i>						

15. *Comments*
 16. *References*
 17. *Notes*

50% of the time. The Marsaglia technique generated samples that, at the 10% level, passed the test on the mean 80% of the time, and on the variance 90% of the time. The Marsaglia technique consistently passed the Kolmogorov-Smirnov test but appeared a little heavy in the 'tails' none the less. For a further examination of this see Addendum 1. The half-Gaussian technique failed the Kolmogorov-Smirnov test once but appeared better behaved in the tails. The graph on page 21 is a scatter diagram for the numbers produced by the Marsaglia technique. The graph consists of 4500 points. Marsaglia's technique produced better results for the first four moments. The decisive factor that leads to the selection of one of the generators is the time to generate each number. The Marsaglia technique is more than twice as fast as the half-Gaussian technique, is the one used in further developments, and is the one recommended for general use.



Scatter Diagram
For The Normal Random Number Generator
(Scale: 1.0 units per inch)



5. The Normal Distribution (Bivariate Case)

5.1 Distribution characteristics.

It is desired to generate pairs of numbers (X_{1i}, X_{2i}) such that the two-dimensional random variable (X_1, X_2) has the joint density function

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp \left[-\frac{1}{2(1-\rho^2)} \left\{ \left(\frac{x_1 - u_1}{\sigma_1} \right)^2 - 2\rho \frac{x_1 - u_1}{\sigma_1} \frac{x_2 - u_2}{\sigma_2} + \left(\frac{x_2 - u_2}{\sigma_2} \right)^2 \right\} \right]$$

for all (x_1, x_2) . The constants are u_1 and u_2 , the means; $\sigma_1 (> 0)$, $\sigma_2 (> 0)$, the standard deviations; and $\rho (-1 \leq \rho \leq 1)$, the correlation coefficient. Thus the requirement for a random vector must be accompanied by the specification of the mean vector (u_1, u_2) , the variances (the squares of the standard deviation), and the correlation coefficient. Another equivalent form of the input would be the mean vector and the covariance matrix. This last form of input will be used in the general multivariate case. The distribution is specified in matrix notation as follows:

$$f(X) = f(x_1, x_2) = \frac{|R|^{1/2}}{(2\pi)^{p/2}} \exp \left\{ -\frac{1}{2} (Y - U)' R (Y - U) \right\}$$

where $f(X)$ is the joint density function of the x 's; p is the dimension-in this case two; U is the mean vector; and R is the inverse of the covariance matrix. Thus $|R|^{1/2}$ is the square root of the determinant of the inverse of the covariance matrix; $(Y - U)' R (Y - U)$ is a quadratic form, where $(Y - U)'$ is a one by p vector, R is a p by p matrix, and $(Y - U)$ is a p by one vector.

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Dear Sirs:

I am writing to you regarding the manuscript of the paper titled "The Role of the ... in the ... of ...". I have received your letter of the ... dated ... and I am pleased to hear that you are interested in the work. The manuscript is currently in the final stages of preparation and I expect to have it ready for submission by the end of the month. I will be sure to provide you with a copy of the final version of the manuscript and the corresponding figures. I am also happy to provide you with a copy of the manuscript in electronic format if you prefer. Please let me know if you have any further comments or questions. I am looking forward to your response.

Sincerely,
[Signature]

5.2 Method of generation.

If $R(J)$ is a normal random number, the random bivariate vector $(VN1, VN2)$ is formed as follows:

$$VN1 = (\sigma_1)R(J) + u_1$$

$$VN2 = (\sigma_1)R(J) + (\sigma_2)R(J+1)(\sqrt{1-\rho^2}) + u_2$$

The source of the normal random numbers is the Marsaglia routine described in the previous section. The generator is Appendix V.

5.3 Testing the generator.

First the maximum likelihood estimators of the mean vector and the covariance matrix are formed [12]. The maximum likelihood estimators for the parameters are:

$$\hat{\mu} = (\bar{x}_1 \quad \bar{x}_2)' = \left(\frac{1}{N} \sum x_{1i} \quad \frac{1}{N} \sum x_{2i} \right)'$$

$$\hat{\sigma}_1^2 = \frac{1}{N} (\sum x_{1i} - \bar{x}_1)^2 = \frac{1}{N} (\sum x_{1i}^2 - N \bar{x}_1^2)$$

$$\hat{\sigma}_2^2 = \frac{1}{N} (\sum x_{2i} - \bar{x}_2)^2 = \frac{1}{N} (\sum x_{2i}^2 - N \bar{x}_2^2)$$

$$\hat{\rho}_{12} = \frac{\sum (x_{1i} - \bar{x}_1)(x_{2i} - \bar{x}_2)}{\sqrt{\sum (x_{1i} - \bar{x}_1)^2 \sum (x_{2i} - \bar{x}_2)^2}}$$

$$\hat{\sigma}_{12} = \sqrt{\hat{\sigma}_1^2 \hat{\sigma}_2^2} \hat{\rho}_{12}$$

The distribution of the mean when the covariance matrix is unknown was shown by Hotelling to be a multivariate analogue of the t-test and is called the generalized T statistic. However, the covariance matrix is known and once again we can use a more powerful test.

$$H_0: u = (U_{10} \ U_{20})' \quad H_A: u \neq (U_{10} \ U_{20})'$$

Construct the statistic H such that:

$$H = N(x_1 - u_{10}, x_2 - u_{20})C^{-1}(x_1 - u_{10}, x_2 - u_{20})'$$

where C is the given covariance matrix. If $H < \chi^2_2(\alpha)$ we accept the null hypothesis. The test for a hypothesized mean vector $U = (0, 0)'$, and a covariance matrix with variances one and correlation coefficient ρ , reduces to calculating H such that:

$$H = \frac{N}{1-\rho^2} (\bar{x}_1 \quad \bar{x}_2) \begin{pmatrix} \frac{1}{\sigma_1^2} & -\frac{\rho}{\sigma_1\sigma_2} \\ -\frac{\rho}{\sigma_1\sigma_2} & \frac{1}{\sigma_2^2} \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix}$$

$$= \frac{N}{1-\rho^2} (\bar{x}_1^2 - 2\rho\bar{x}_1\bar{x}_2 + \bar{x}_2^2), \text{ for } \sigma_1 = \sigma_2 = 1.$$

for sample sizes of 1000 the test will be made for $\rho = 0.25, 0.5, 0.75$. At the 10% level if $H \leq 4.61$ accept the hypothesis. In order to test whether the covariance matrix is a given matrix, the test as outlined in Anderson [12] could be used, but before this a digression into the philosophy of testing is in order.

The purpose of these hypothesis tests is in general to render a judgement as to whether a sample of data from some type of experiment is distributed according to a certain distribution with certain parameters. This type of test was applicable to our uniform random number generator, but its use seems extraneous for the distributions derived from this generator. The transformations from the uniform to the normal to bivariate normal are either theoretically exact or controllably close to being exact. What is really needed is a method to ensure that any 'inaccuracies' there may have been in the basic generator are not somehow amplified in the transformation so as to bias the derived distribution.

the present state of the world, and the progress of the human mind, it is not possible to see the

same things in the same light as we did when we first began to see them.

And it is not only the progress of the human mind, but the progress of the human race, which

has made it possible for us to see the same things in the same light.

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To this end sophisticated testing procedures do not seem to be in order. Instead the testing will be restricted to calculation of estimators, and some goodness of fit tests. The testing plan for the bivariate generator is to test 10 sets of 1000 each for each of several correlation coefficients. The generator was also tested for some odd combinations of parameters- such as $U=(100.0, -100.0)$, $\sigma_1^2=0.5$, $\sigma_2^2=25.0$, $\rho = -0.8$.

With 0.75 as the correlation coefficient, the fourth set showed a disagreeably large difference in the maximum likelihood estimators. The H statistic was close to the critical value at the 10% level. The sample passed the test in 90% of the cases. With 0.50 as the correlation coefficient the sample passed the H test 90% of the time. Sample set four once again had rather low covariance estimators and once again had 3.76 as the value for H (compared with 4.61 for the critical value). Sample set six was again the culprit in not passing the H test. A similar result was noted in the set using 0.25 as a correlation coefficient. This suggests the advisability of further testing of the Marsaglia generator in these ranges. Complete test results are in Appendix VI. The generator performs as expected and is recommended for use.

6. The Normal Distribution (Multivariate Case).

6.1 Distribution characteristics.

As is noted in Section 6.1 the desired distribution is

$$f(X) = f(x_1, x_2, \dots, x_N) = \frac{|R|^{1/2}}{(2\pi)^{N/2}} \exp\left(-\frac{1}{2}(Y-U)'R(Y-U)\right)$$

where R is the inverse of the covariance matrix.

6.2 Method of generation.

As shown by Wold [16], the method is based on a triangularization of the covariance matrix such that if:

$$\begin{bmatrix} C_{11} & C_{12} & \dots & C_{1N} \\ C_{21} & C_{22} & \dots & C_{2N} \\ \vdots & & \ddots & \vdots \\ C_{N1} & C_{N2} & \dots & C_{NN} \end{bmatrix} = \begin{bmatrix} P_{11} & 0 & \dots & 0 \\ P_{21} & P_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ P_{N1} & P_{N2} & \dots & P_{NN} \end{bmatrix} \begin{bmatrix} P_{11} & P_{21} & \dots & P_{N1} \\ 0 & P_{22} & \dots & P_{N2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & 0 & P_{NN} \end{bmatrix}$$

then the N-dimensional random vector Z may be formed as follows.

The RN(I) are the normal random numbers generated by the Marsaglia technique.

$$Z(1) = P_{11} \times RN(1)$$

$$Z(2) = P_{21} \times RN(1) + P_{22} \times RN(2)$$

$$Z(3) = P_{31} \times RN(1) + P_{32} \times RN(2) + P_{33} \times RN(3)$$

$$\vdots$$

$$Z(N) = P_{N1} \times RN(1) + P_{N2} \times RN(2) + \dots + P_{NN} \times RN(N).$$

The method as programmed, assumes all the means are zero, but simple addition of the mean when required will remedy this. The matrix triangularization is based on a symmetric, positive definite or positive semi-definite matrix C. Thus any pair of the random variables can have a partial correlation coefficient of one. The routine will set all elements of the vector that are dependent equal to zero. This procedure was selected since in order to relate the

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variables properly requires an inordinately extra amount of computer time- time that even when not needed adds to execution time. If the user desires some variable to be a linear transformation of some others, then he only needs to keep track of which variables these are, and where the program sets the vector element equal to zero, substitute the appropriate linear combination of the corresponding independent elements of the vector. The routine also checks and where the dependence is very close to one will assume a correlation of one, and proceed as noted above. This procedure is required to prevent division by numbers very close to zero. The following example will clarify the above explanation. Suppose it is desired to generate random vectors from a distribution with mean vector one and covariance matrix C, where:

$$C = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 4 & 5 \\ 3 & 5 & 8 \end{bmatrix}$$

Thus column three is a linear combination, in fact the sum, of columns one and two. This means that the third variable is not an independent variable. The triangulation routine will produce a matrix P of the form:

$$P = \begin{bmatrix} a & o & o \\ b & d & o \\ c & e & o \end{bmatrix}$$

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As expected column three is identically zero. Thus if the first two normal (0,1) numbers generated are denoted by R1 and R2, the generated vector will be of the form:

$$Z = (aR1, \quad bR1 + dR2, \quad 0) \quad .$$

Since it has been determined that the third variable is the sum of the first two and also that we desire all the means to be one then the desired vector is of the form:

$$Z = (aR1 + 1, \quad bR1 + dR2 + 1, \quad aR1 + bR1 + dR2 + 1)$$

The generator is Appendix VII.

6.3 Testing the generator.

The maximum likelihood estimators of the mean vector and the covariance matrix are formed as follows:

$$BML(J) = \frac{1}{M} \sum_{I=1}^M Y(I,J) \quad \text{For } J=1, 2, \dots, N$$

$$C(I,J) = \frac{1}{M} \left[\sum_{K=1}^M \{Y(I,K) \cdot Y(J,K)\} \right] \quad \text{for } I, J=1, 2, \dots, N.$$

N is the dimension of the covariance matrix, M is the number of sample vectors, the Y(I,J) are the vector elements, the BML(J) are the elements of the mean vector estimates, and the C(I,J) are the elements of the covariance matrix estimate. The results of the tests for various covariance matrices are listed in Appendix VIII.

7. The Poisson Distribution.

7.1 Distribution characteristics.

It is desired to generate numbers such that:

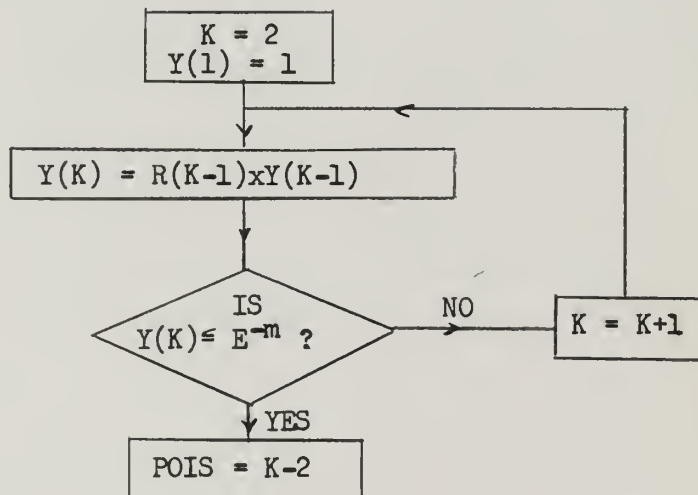
$$P [x = a] = e^{-m} \frac{m^a}{a!} \quad \text{for } m > 0 ; a = a_1, \dots$$

Thus the Poisson distribution is a discrete distribution for integer values of a , and is characterized by the parameter m .

The first four moments are $M_1 = M_2 = M_3 = m$, and $M_4 = 3m^2 + m$.

7.2 Method of generation.

This method is due to Kahn [11] and is a theoretically exact transformation. The flow chart of the routine is shown below:



POIS is the generated Poisson random number, m is the parameter, E is the irrational number 2.71828...

7.3 Testing the generator.

The first four moments are calculated for samples of 5000 or 1000 with the parameter m taking on various values. The Kolmogorov-Smirnov goodness of fit test is applied to some of the

samples. As Tate and Clelland [19] have stated the test is applicable to discrete distributions with negligible changes in significance for large sample sizes. The generator performed well and consistently passed the test. The test results are in Appendix X, while the generator itself is in Appendix IX.

8. The Exponential Distribution.

8.1 Distribution characteristics.

It is desired to generate random numbers such that the density function is: $f(a) = e^{-a}$.

The first four moments are $M_1 = M_2 = 1$, $M_3 = 2$, $M_4 = 9$. The distribution is often specified as:

$$f(a) = \lambda e^{-\lambda a} \quad \text{for } \lambda > 0, a \geq 0$$

where λ is the parameter of the distribution. The generator here takes the case where $\lambda = 1$. However the exponential distribution has the characteristic that if the numbers generated here, (EXPL) for which $\lambda = 1$, are simply multiplied by the parameter desired for the distribution (LAMBDA), then the desired numbers are generated.

$$\text{EXPL} = \text{EXPL} * \text{LAMBDA}$$

8.2 Method of generation.

This method is a theoretically exact procedure due to Marsaglia [13]. A more obvious method would be to take the integral transformation- the negative logarithm of a uniform random number. However this method is slower on most computers than the one demonstrated here. Let $C = 1/(e-1)$, and let the random variable N take on the values 1,2,3,4,... with probabilities $c, c/2!, c/3!, \dots$. Then let the random variable M take values 0,1,2,3,... with probabilities $1/ce, 1/ce^2, 1/ce^3, \dots$. Then we form the desired random number:

$$\text{EXPL} = M + \text{MIN}(U_1, U_2, \dots, U_N)$$

The CODAP generating routine is Appendix XI. The sample moments and the results of the goodness of fit tests are in Appendix XII.

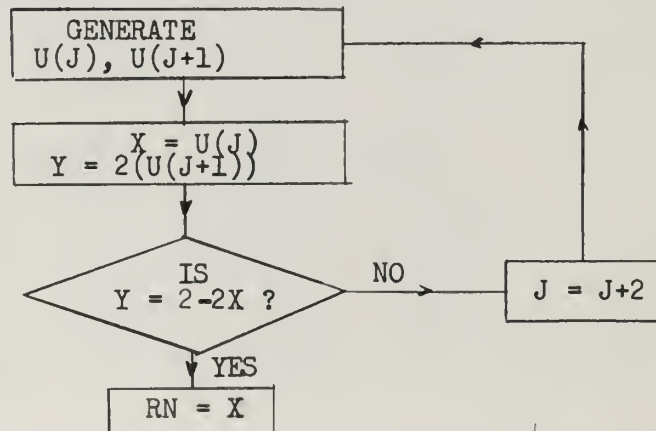
9. A General Disbribution.

The method of obtaining random numbers described in this section is applicable only to a restricted set of distributions. However, the method is applicable to a type of distribution that frequently occurs in model building and war gaming. The method is examined by use of an example. A discussion of how and when to use this method for other distributions is included.

It is desired to produce numbers from the density function $f(x)$ such that:

$$f(x) = 2-2x \quad \text{for } 0 \leq x \leq 1$$

The method is based on drawing uniform numbers in pairs, normalizing the scale of the numbers, and testing to see if the point formed by the pair lies under the curve described by the density function. If it does, the x coordinate of the point is taken as the random number; if it does not, another pair of uniform numbers are drawn and the process is continued. The flow chart for $f(x) = 2-2x$ is:



The $U(J)$ are the uniform $(0,1)$ random numbers and RN is the random

number from the distribution $f(X)$. Thus the method could be applied with theoretical exactness to any bounded continuous distribution. Another commonly used density is $g(x)$ where

$$g(x) = nx^{n-1} \quad \text{for } 0 < x < 1, \quad n \geq 1$$

This density function is bounded and continuous and a member of the class to which this method is applicable. In many applications the user must make a judgement about the amount of use a generator is going to get. If it is intended for heavy use it may be well to explore the literature for, or to design, a more efficient generator than the type described in this section. However, if the generator is to only be used a limited amount, or if only a restricted amount of resources are available, this generator is easily programmed and will be eminently satisfactory in a wide variety of cases. The efficiency of the generator may be defined as the reciprocal of the expected number of iterations needed to produce a point under the curve.

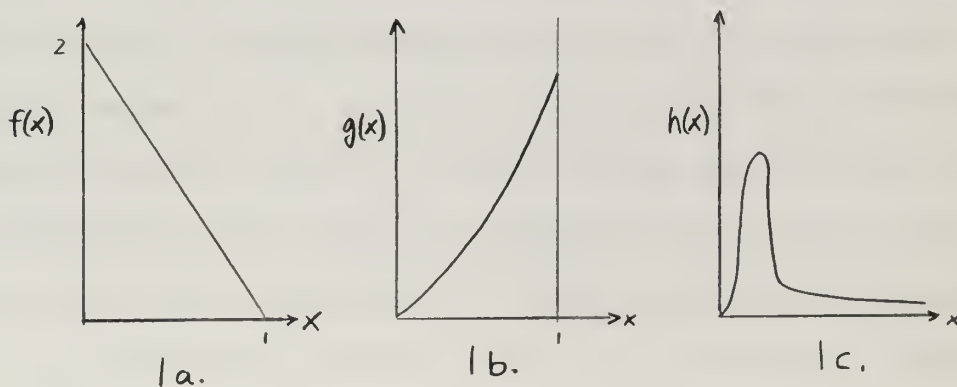
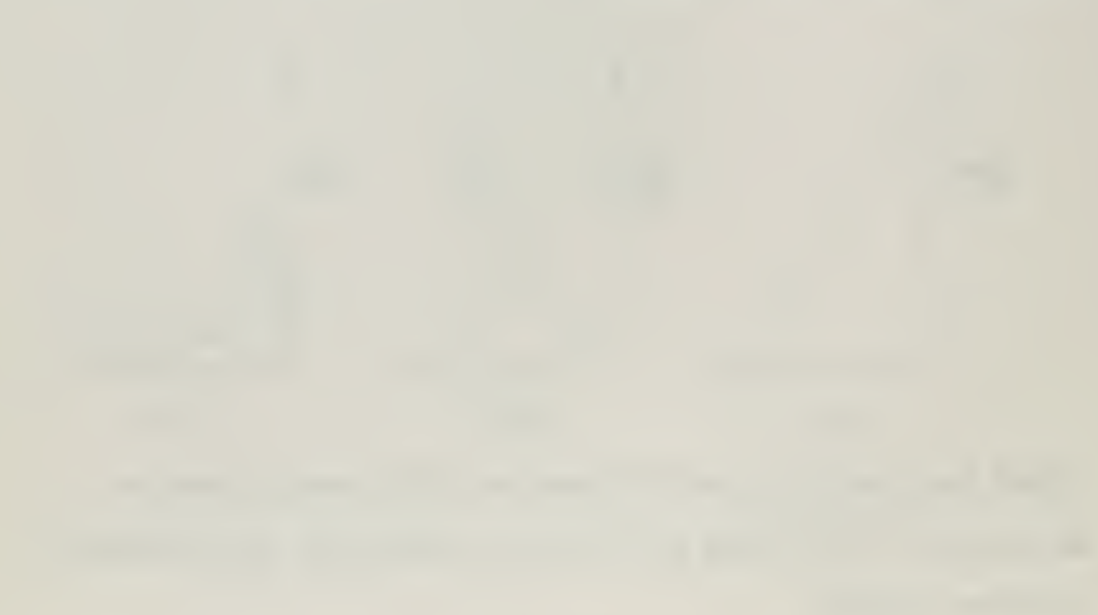
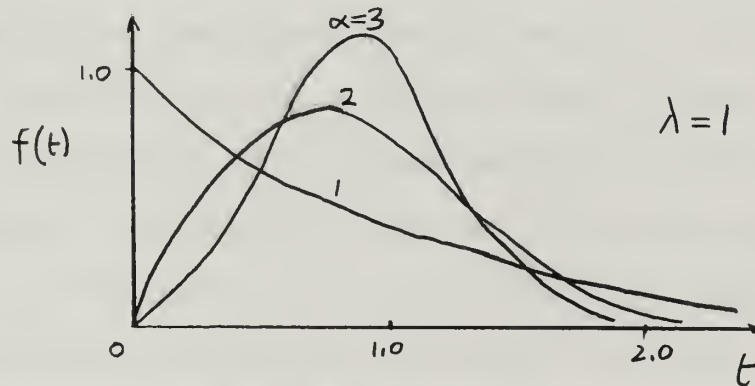


Figure 1a clearly has an efficiency of $1/2$, figure 1b has an efficiency of less than $1/2$, and as n gets large the efficiency decreases rapidly.

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A distribution such as figure 1c where $h(x)$ goes to zero at some large value of x may vary well have such a low efficiency as to make this generator unsuitable. If $h(x)$ only approaches zero as x becomes large, and therefore x is not bounded, then another generator should be used. However, if one cannot be found and the user is not too concerned with the tail of the distribution, we can easily adapt this generator. Suppose we wished to generate numbers for the Weibull distribution, which is a three parameter distribution used widely in reliability theory, then the following procedure might be followed:



Since the distribution may take any of the forms in figure 2, we must first limit the development to those parameter combinations that are bounded at $x = 0$. Since the user is also often concerned with the behaviour in part of the tail it must first be decided if the distribution could simply be truncated at some point. Even if this is acceptable an efficiency of .00001 can easily be envisioned. If it is unacceptable, the tail beyond some point could be closely approximated by the exponential distribution.

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The graph shows the relationship between the two variables over time. The straight line represents a constant rate of change, while the curve represents a variable rate of change. The curve starts at the origin, reaches a maximum value of 5 at time 5, and then decreases to 0 at time 10. This indicates that the rate of change is highest at time 5 and lowest at time 0 and time 10.

10. Summary and Conclusions.

The analyst who desires to use any of the generators demonstrated here is once again warned that although the generators are recommended for general use they all have aberrations of some kind. The general form of these aberrations is noted where known. If a user suspects from analysis of his results that the aberration in the generator is influencing his results then it is recommended that he first modify the uniform generator by one of the methods suggested in section two. The chance of this occurring is remote and other parts of the model should be checked carefully.

The programs demonstrated here are very fast. The problem of speed is stressed here and is a major decision criterion in the selection of generators. In some applications the speed may not be as important a factor, and in this case the type of generator discussed in section nine may be very cost effective in that little investment in programming and testing is required. The problem of measuring the speed of generation of a number is not as simple as it may appear. The generation time depends on the program using the generator and also on the method used to time the routine. The time to generate a number based on the Control Data Corporation specifications for the 1604 computer theoretically should be 121 microseconds. The observed generation times vary from 370 to 700 microseconds.

The tests used here are statistically sound, but the meaning of the results of several tests of the same generated sample could bear further investigation.

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Some very interesting new methods for generating random numbers from various distributions have been developed by H. Rubin. Rubin's work is soon to appear as a Stanford Applied Statistics Laboratory Technical Report. It would be of interest to compare his methods with the technique used above.

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Appendix I.

The Fortran 63 CODAP function subprogram for generating uniform (0,1) random numbers.

The program is called by using a variable, 'UNIFORM(DUMMY)'. The argument 'DUMMY' is not used. For example: A = UNIFORM(DUMMY)+ B. No common or dimension entries are required.

The observed average length of the program is 552 microseconds.

The starting number may be change to 'R OCT 5402033450727422' if it is desired to enter the generator near the middle of the period.

	IDENT	UNIFORM	
	ENTRY	UNIFORM	
M1	OCT	4000000000000000	
M2	OCT	2000000000000000	
R	OCT	1777777777777777	
UNIFORM	SLJ	**	
	LDA	*	
	ALS	24	
	INA	1	
	SAU	EXIT	
	ENQ	0	
	LDA	R	
	LLS	+19	
	SCL	M1	
	ADD	R	
	ADD	R	
	ADD	R	
	STA	R	
	ARS	11	
	ADD	M2	
	FAD	M2	
EXIT	SLJ	**	
	END		

$-(2^{46}-1)$

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Appendix II.

The half-Gaussian technique for producing normal random numbers.

The number is generated by the use of the expression 'GSRN(BLK)'.

The number is returned as GSRN. The argument 'BLK' is not used.

No external calls are made.

The observed average length of time to generate one number is

3625 microseconds.

	IDENT	GSRN
	ENTRY	GSRN
GSRN	SLJ	**
	LDA	*
	ALS	24
	INA	1
	SAU	GSRN
.190	RTJ	UNIFORM
	STA	B1
	FAD	=020014000000000000
	STA	TEMP
	LDA	B1
	FSB	=020014000000000000
	FDV	TEMP
	STA	X
	FMU	X
	STA	X2
	FMU	=01774707070705726
	FAD	=01775444444443023
	FMU	X2
	FAD	=01775631463146315
	FMU	X2
	FAD	=01776525252524341
	FMU	X2
	FAD	=020014000000000000
	FMU	X
	FMU	=0577537777777777
	STA	Y
	RTJ	UNIFORM
	STA	B1
	FAD	=020014000000000000
	STA	TEMP
	LDA	B1
	FSB	=020014000000000000
	FDV	TEMP

The first part of the report discusses the general situation of the country and the progress of the work. It also mentions the results of the various committees and the work of the various departments. The second part of the report discusses the results of the various committees and the work of the various departments. The third part of the report discusses the results of the various committees and the work of the various departments.

Table 1	
Year	Amount
1911	1000
1912	1200
1913	1500
1914	1800
1915	2000
1916	2200
1917	2500
1918	2800
1919	3000
1920	3200
1921	3500
1922	3800
1923	4000
1924	4200
1925	4500
1926	4800
1927	5000
1928	5200
1929	5500
1930	5800
1931	6000
1932	6200
1933	6500
1934	6800
1935	7000
1936	7200
1937	7500
1938	7800
1939	8000
1940	8200
1941	8500
1942	8800
1943	9000
1944	9200
1945	9500
1946	9800
1947	10000
1948	10200
1949	10500
1950	10800
1951	11000
1952	11200
1953	11500
1954	11800
1955	12000
1956	12200
1957	12500
1958	12800
1959	13000
1960	13200
1961	13500
1962	13800
1963	14000
1964	14200
1965	14500
1966	14800
1967	15000
1968	15200
1969	15500
1970	15800
1971	16000
1972	16200
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1979	18000
1980	18200
1981	18500
1982	18800
1983	19000
1984	19200
1985	19500
1986	19800
1987	20000
1988	20200
1989	20500
1990	20800
1991	21000
1992	21200
1993	21500
1994	21800
1995	22000
1996	22200
1997	22500
1998	22800
1999	23000
2000	23200
2001	23500
2002	23800
2003	24000
2004	24200
2005	24500
2006	24800
2007	25000
2008	25200
2009	25500
2010	25800
2011	26000
2012	26200
2013	26500
2014	26800
2015	27000
2016	27200
2017	27500
2018	27800
2019	28000
2020	28200
2021	28500
2022	28800
2023	29000
2024	29200
2025	29500
2026	29800
2027	30000
2028	30200
2029	30500
2030	30800
2031	31000
2032	31200
2033	31500
2034	31800
2035	32000
2036	32200
2037	32500
2038	32800
2039	33000
2040	33200
2041	33500
2042	33800
2043	34000
2044	34200
2045	34500
2046	34800
2047	35000
2048	35200
2049	35500
2050	35800
2051	36000
2052	36200
2053	36500
2054	36800
2055	37000
2056	37200
2057	37500
2058	37800
2059	38000
2060	38200
2061	38500
2062	38800
2063	39000
2064	39200
2065	39500
2066	39800
2067	40000
2068	40200
2069	40500
2070	40800
2071	41000
2072	41200
2073	41500
2074	41800
2075	42000
2076	42200
2077	42500
2078	42800
2079	43000
2080	43200
2081	43500
2082	43800
2083	44000
2084	44200
2085	44500
2086	44800
2087	45000
2088	45200
2089	45500
2090	45800
2091	46000
2092	46200
2093	46500
2094	46800
2095	47000
2096	47200
2097	47500
2098	47800
2099	48000
2100	48200
2101	48500
2102	48800
2103	49000
2104	49200
2105	49500
2106	49800
2107	50000
2108	50200
2109	50500
2110	50800
2111	51000
2112	51200
2113	51500
2114	51800
2115	52000
2116	52200
2117	52500
2118	52800
2119	53000
2120	53200
2121	53500
2122	53800
2123	54000
2124	54200
2125	54500
2126	54800
2127	55000
2128	55200
2129	55500
2130	55800
2131	56000
2132	56200
2133	56500
2134	56800
2135	57000
2136	57200
2137	57500
2138	57800
2139	58000
2140	58200
2141	58500
2142	58800
2143	59000
2144	59200
2145	59500
2146	59800
2147	60000
2148	60200
2149	60500
2150	60800
2151	61000
2152	61200
2153	61500
2154	61800
2155	62000
2156	62200
2157	62500
2158	62800
2159	63000
2160	63200
2161	63500
2162	63800
2163	64000
2164	64200
2165	64500
2166	64800
2167	65000
2168	65200
2169	65500
2170	65800
2171	66000
2172	66200
2173	66500
2174	66800
2175	67000
2176	67200
2177	67500
2178	67800
2179	68000
2180	68200
2181	68500
2182	68800
2183	69000
2184	69200
2185	69500
2186	69800
2187	70000
2188	70200
2189	70500
2190	70800
2191	71000
2192	71200
2193	71500
2194	71800
2195	72000
2196	72200
2197	72500
2198	72800
2199	73000
2200	73200
2201	73500
2202	73800
2203	74000
2204	74200
2205	74500
2206	74800
2207	75000
2208	75200
2209	75500
2210	75800
2211	76000
2212	76200
2213	76500
2214	76800
2215	77000
2216	77200
2217	77500
2218	77800
2219	78000
2220	78200
2221	78500
2222	78800
2223	79000
2224	79200
2225	79500
2226	79800
2227	80000
2228	80200
2229	80500
2230	80800
2231	81000
2232	81200
2233	81500
2234	81800
2235	82000
2236	82200
2237	82500
2238	82800
2239	83000
2240	83200
2241	83500
2242	83800
2243	84000
2244	84200
2245	84500
2246	84800
2247	85000
2248	85200
2249	85500
2250	85800
2251	86000
2252	86200
2253	86500
2254	86800
2255	87000
2256	87200
2257	87500
2258	87800
2259	88000
2260	88200
2261	88500
2262	88800
2263	89000
2264	89200
2265	89500
2266	89800
2267	90000
2268	90200
2269	90500
2270	90800
2271	91000
2272	91200
2273	91500
2274	91800
2275	92000
2276	92200
2277	92500
2278	92800
2279	93000
2280	93200
2281	93500
2282	93800
2283	94000
2284	94200
2285	94500
2286	94800
2287	95000
2288	95200
2289	95500
2290	95800
2291	96000
2292	96200
2293	96500
2294	96800
2295	97000
2296	97200
2297	97500
2298	97800
2299	98000
2300	98200
2301	98500
2302	98800
2303	99000
2304	99200
2305	99500
2306	99800
2307	100000
2308	100200
2309	100500
2310	100800
2311	101000
2312	101200
2313	101500
2314	101800
2315	102000
2316	102200
2317	102500
2318	102800
2319	103000
2320	103200
2321	103500
2322	103800
2323	104000
2324	104200
2325	104500
2326	104800
2327	105000
2328	105200
2329	105500
2330	105800
2331	106000
2332	106200
2333	106500
2334	106800
2335	107000
2336	107200
2337	107500
2338	107800
2339	108000
2340	108200
2341	108500
2342	108800
2343	109000
2344	109200
2345	109500
2346	109800
2347	110000
2348	110200
2349	110500
2350	110800
2351	111000
2352	111200
2353	111500
2354	111800
2355	112000
2356	112200
2357	112500
2358	112800
2359	113000
2360	113200
2361	113500
2362	113800
2363	114000
2364	114200
2365	114500
2366	114800
2367	115000
2368	115200
2369	115500
2370	115800
2371	116000
2372	116200
2373	116500
2374	116800
2375	117000
2376	117200
2377	117500
2378	117800
2379	118000
2380	118200
2381	118500
2382	118800
2383	119000
2384	119200
2385	119500
2386	119800
2387	120000
2388	120200
2389	120500
2390	120800
2391	121000
2392	121200
2393	121500
2394	121800
2395	122000
2396	122200
2397	122500
2398	122800
2399	123000
2400	123200
2401	123500
2402	123800
2403	124000
2404	124200
2405	124500
2406	124800
2407	125000
2408	125200
2409	125500
2410	125800
2411	126000
2412	126200
2413	126500
2414	126800
2415	127000
2416	127200
2417	127500
2418	127800
2419	128000
2420	128200
2421	128500
2422	128800
2423	129000
2424	129200
2425	129500
2426	129800
2427	130000
2428	130200
2429	130500
2430	130800
2431	131000
2432	131200
2433	131500
2434	131800
2435	13200

	STA	X
	FMU	X
	STA	X2
	FMU	=01774707070705726
	FAD	=01775444444443023
	FMU	X2
	FAD	=01775631463146315
	FMU	X2
	FAD	=01776525252524341
	FMU	X2
	FAD	=02001400000000000
	FMU	X
	FMU	=05774377777777777
	STA	Z
	LDA	Y
	FSB	=02001400000000000
	STA	YML
	FMU	YML
	THS	Z
	SLJ	.190
	RTJ	UNIFORM
	THS	=02000400000000000
	SLJ	*+2
	LDA	Y
	SLJ	GSRN
UNIFORM	LAC	Y
	SLJ	GSRN
	SLJ	**
	ENQ	O
	LDA	R
	LLS	19
	SLC	=04000000000000000
	ADD	R
	ADD	R
	ADD	R
	STA	R
	ARS	11
	ADD	=02000000000000000
	FAD	=02000000000000000
	SLJ	UNIFORM
R	OCT	17777777777777777
B1	BSS	1
TEMP	BSS	1
X	BSS	1
Y	BSS	1
YML	BSS	1
Z	BSS	1
X2	BSS	1
X3	BSS	1
X5	BSS	1
X7	BSS	1
X9	BSS	1
	END	

Appendix III.

The number is generated by use of the variable 'MARS(ZQ)'. The argument 'ZQ' is not used. For example: 'A = MARS(ZQ)/9.

External calls are made to LOGF, SQRTF, and EXPF.

No common dimension entries are required.

The observed average length of time to generate one number is 1503 microseconds.

	IDENT	ZMARS
	ENTRY	ZMARS.
UNIFORM	SLJ	**
	ENQ	0
	LDA	R
	LLS	19
	SCL	=040000000000000000
	ADD	R
	ADD	R
	ADD	R
	STA	R
	ARS	11
	ADD	=020000000000000000
	FAD	=020000000000000000
	SLJ	UNIFORM
R	OCT	1777777777777777
ZMARS	SLJ	**
	LDA	*
	ALS	24
	INA	1
	SAU	Z MARS
	RTJ	UNIFORM
G1	THS	P1
	SLJ	G2
	RTJ	UNIFORM
	STA	NORM
	RTJ	UNIFORM
	FAD	NORM
	STA	NORM
	RTJ	UNIFORM
	FAD	NORM
	FSB	=020016000000000000
	STA	NORM
	FAD	NORM
	SLJ	ZMARS

G2	THS	P2
	SLJ	G3
	RTJ	UNIFORM
	STA	NORM
	RTJ	UNIFORM
	FAD	NORM
	FSB	=020014000000000000
	FMU	=020016000000000000
	SLJ	ZMARS
G3	THS	P3
	SLJ	G4
G31	RTJ	UNIFORM
	FMU	=020046000000000000
	FSB	=020026000000000000
	STA	T
	AJP	2 GSPOS
	SCM	=0777777777777777
G3POS	STA	T+1
	FMU	T+1
	STA	T+2
	FMU	=0577737777777777
	CALL	EXPF
	FMU	C1
	STA	T+3
	LDA	T+1
	THS	=020014000000000000
	SLJ	T1.5
	LDA	=020026000000000000
	FSB	T+2
	FMU	C2
	FAD	T+3
	STA	T+3
	LDA	=020016000000000000
	FSB	T+1
	FMU	C3
	FAD	T+3
	SLJ	G3END
T1.5	THS	=020016000000000000
	SLJ	T3.0
	LDA	=020026000000000000
	FSB	T+1
	STA	T+4
	FMU	T+4
	FMU	C4
	FAD	T+3
	STA	T+3
	LDA	=020016000000000000
	FSB	T+1
	FMU	C3
	FAD	T+3
	SLJ	G3END

T3.0	LDA	=020026000000000000
	FSB	T+1
	STA	T+4
	FMU	T+4
	FMU	C4
	FAD	T+3
G3END	STA	T+4
	RTJ	UNIFORM
	FMU	C5
	THS	T+4
	SLJ	G31
	LDA	T
	SLJ	ZMARS
G4	RTJ	UNIFORM
	FMU	=020024000000000000
	FSB	=020014000000000000
	STA	T
	FMU	T
	STA	T+1
	RTJ	UNIFORM
	FMU	=020024000000000000
	FSB	=020014000000000000
	STA	T+2
	FMU	T+2
	FAD	T+1
	THS	=020014000000000000
	SLJ	G4
	STA	T+1
	CALL	LOGF
	STA	T+3
	FAD	T+3
	SCM	=077777777777777777
	FAD	=020044400000000000
	FDV	T+1
	CALL	SQRTF
	STA	T+3
	FMU	T
	STA	T+4
	AJP	2 *+1
	SCM	=077777777777777777
	THS	=020026000000000000
	SLJ	GOOD
	LDA	T+3
	FMU	T+2
	STA	T+4
	AJP	2 *+1
	SCM	=077777777777777777
	THS	=020026000000000000
	SLJ	GOOD

	SLJ	G4
GOOD	LDA	T+4
	SLJ	ZMARS
C1	DEC	17.49731196
C2	DEC	-4.73570326
C3	DEC	-2.15787544
C4	DEC	-2.36785163
C5	DEC	0.358
P1	DEC	0.8638
P2	DEC	0.9745
P3	DEC	0.9973002039
T	BSS	5
NORM	OCT	0
	END	

Appendix IV.

Test results for Marsaglia normal generator.

The test consists of ten consecutive samples with 10000 numbers in each.

MEAN	M2	M3	M4	$\sqrt{N}\bar{X}$	$\frac{S-(N-1)}{\sqrt{2(N-1)}}$	d/n
.0	1.0	.0	3.0	-1.645 to 1.645	-1.645 to 1.645	
-.02474	.99844	-.04435	2.96870	-2.4737	-.1104	0.0095
-.01062	.98869	-.07487	2.92989	-1.0625	-.7996	0.0049
-.00564	.99388	-.07973	2.93538	-.5640	-.4330	0.0027
-.00873	1.00452	-.09225	2.99577	-.8729	.3196	0.0060
-.00921	.99945	-.04774	3.03286	-.9209	-.0392	0.0040
-.00683	.99709	-.10706	3.08487	-.6834	-.2057	0.0048
.00028	.99963	-.04737	3.03274	.0277	-.0264	0.0030
.00732	1.01508	-.04782	3.16472	.7317	1.0664	0.0041
-.00410	1.02454	-.05629	3.09820	-.4104	1.7352	0.0050
-.01732	.97325	-.05067	2.79351	-1.7319	-1.8911	0.0050

Appendix IV. (Cont'd)

Test results for half-Gaussian method.

The test consists of ten consecutive samples with 10000 numbers in each.

MEAN	M2	M3	M4	$\sqrt{N}\bar{X}$	$\frac{S-(N-1)}{\sqrt{2(N-1)}}$	d/n
.0	1.0	.0	3.0	-1.645 to 1.645	-1.645 to 1.645	
-.00961	1.03629	-.01728	3.19774	-.9614	2.5661	0.0073
.00301	1.03418	.00973	3.12078	-.3008	.9529	0.0033
-.01107	1.02600	-.01559	3.12095	-1.1066	1.8386	0.0056
.00844	1.04205	-.02587	3.23017	.8443	2.9735	0.0072
.00230	1.01410	-.00458	3.13768	.2295	.9969	0.0044
-.00475	1.02696	-.00209	3.27642	-.4747	1.9064	0.0036
-.02246	1.04439	.01642	3.29951	-2.2459	3.1384	0.0149
-.00372	1.02968	-.01805	3.13442	-.3716	2.0986	0.0065
-.00684	1.04955	.04399	3.28683	-.6841	3.5036	0.0076
.01761	1.01793	-.00633	3.11080	1.7614	1.2680	0.0084

Appendix V.

The bivariate normal random number generator (using Marsaglia's technique).

The number is generated by a call 'CALL AKHN(VN1, VN2)'. The bivariate vector is returned as the arguments VN1, VN2.

Subroutine AKHN has the following arguments in common: SIG1SQ, SIG2SQ, RHO, U1, U2. The SIG1SQ are the desired variances, RHO is the correlation coefficient, and U1 and U2 are the desired means. Thus besides reading in these values in the main program they must be communicated to the subroutine by a common statement such as: 'COMMON/SIG1SQ/SIG1SQ/SIG2SQ/SIG2SQ/RHO/RHO/U1/U1/U2/U2'.

External calls are made to LOGF, SQRTF, and EXPF.

The observed average generation time for a pair of numbers is

6480 microseconds.

	IDENT	AKHN
SIG1SQ	BLOCK	1
	COMMON	SIG1SQ(1)
SIG2SQ	BLOCK	1
	COMMON	SIG2SQ(1)
RHO	BLOCK	1
	COMMON	RHO(1)
U1	BLOCK	1
	COMMON	U1(1)
U2	BLOCK	1
	COMMON	U2(1)
	ENTRY	AKHN
AKHN	SLJ	**
	LDA	*
	ALS	24
	SAU	*+2
+	INA	1
	SAU	EXIT
+	LDA	**
	SAL	VN2

	ALS	24
	SAL	VN1
+	RTJ	ZMARS
+	STA	A1
	LDA	SIG1SQ
+	CALL	SQRTF
+	FMU	A1
	STA	A2
VN1	FAD	U1
	STA	VN1
	LDA	A2
	FMU	RHO
	FAD	U2
	STA	A2
	LAC	RHO
	FMU	RHO
	FAD	=020014000000000000
	FMU	SIG2SQ
+	CALL	SQRTF
+	STA	A1
+	RTJ	ZMARS
+	FMU	A1
VN2	FAD	A2
	STA	VN2
EXIT	SLJ	**
ZMARS	SLJ	**
	RTJ	UNIFORM
G1	THS	P1
	SLJ	G2
	RTJ	UNIFORM
+	STA	NORM
	RTJ	UNIFORM
+	FAD	NORM
	STA	NORM
	RTJ	UNIFORM
+	FAD	NORM
	FSB	=020016000000000000
	STA	NORM
	FAD	NORM
	SLJ	ZMARS
G2	THS	P2
	SLJ	G3
	RTJ	UNIFORM
+	STA	NORM
	RTJ	UNIFORM
+	FAD	NORM
	FSB	=020014000000000000
	FMU	=020016000000000000

	SLJ	ZMARS
G3	THS	P3
	SLJ	G4
G31	RTJ	UNIFORM
+	FMU	=020046000000000000
	FSB	=020026000000000000
	STA	T
	AJP	2 G3POS
	SCM	=077777777777777777
G3POS	STA	T+1
	FMU	T+1
	STA	T+2
	FMU	=057773777777777777
	CALL	EXPF
+	FMU	C1
	STA	T+3
	LDA	T+1
+	THS	=020014000000000000
	SLJ	T1.5
+	LDA	=020026000000000000
	FSB	T+2
	FMU	C2
	FAD	T+3
	STA	T+3
	LDA	=020016000000000000
	FSB	T+1
	FMU	C3
	FAD	T+3
	SLJ	G3END
T1.5	THS	=020016000000000000
	SLJ	T3.0
+	LDA	=020026000000000000
	FSB	T+1
	STA	T+4
	FMU	T+4
	FMU	C4
	FAD	T+3
	STA	T+3
	LDA	=020016000000000000
	FSB	T+1
	FMU	C3
	FAD	T+3
	SLJ	G3END
T3.0	LDA	=020026000000000000
	FSB	T+1
	STA	T+4
	FMU	T+4
	FMU	C4
	FAD	T+3

G3END	STA	T+4
	RTJ	UNIFORM
+	FMU	C5
+	THS	T+4
	SLJ	G31
+	LDA	T
	SLJ	ZMARS
G4	RTJ	UNIFORM
+	FMU	=020024000000000000
	FSB	=020014000000000000
	STA	T
	FMU	T
	STA	T+1
	RTJ	UNIFORM
+	FMU	=020024000000000000
	FSB	=020014000000000000
	STA	T+2
	FMU	T+2
	FAD	T+1
+	THS	=020014000000000000
	SLJ	G4
	STA	T+1
	CALL	LOGF
+	STA	T+3
	FAD	T+3
	SCM	=077777777777777777
	FAD	=020044400000000000
	FDV	T+1
	CALL	SQRTF
+	STA	T+3
	FMU	T
	STA	T+4
	AJP	2 *+1
	SCM	=077777777777777777
+	THS	=020026000000000000
	SLJ	GOOD
+	LDA	T+3
	FMU	T+2
	STA	T+4
	AJP	2 *+1
	SCM	=077777777777777777
+	THS	=020026000000000000
	SLJ	GOOD
+	SLJ	G4
GOOD	LDA	T+4
	SLJ	ZMARS
UNIFORM	SLJ	**
	ENQ	O
	LDA	R

	LLS	19
	SCL	=040000000000000000
	ADD	R
	ADD	R
	ADD	R
	STA	R
	ARS	11
	ADD	=020000000000000000
	FAD	=020000000000000000
	SLJ	UNIFORM
C1	DEC	17.49731196
C2	DEC	-4.73570326
C3	DEC	-2.15787544
C4	DEC	-2.36785163
C5	DEC	0.358
P1	DEC	0.8638
P2	DEC	0.9745
P3	DEC	0.9973002039
T	BSS	5
NORM	OCT	0
R	OCT	1777777777777777
A1	BSS	1
A2	BSS	1
	END	

Appendix VI.

Test results for bivariate normal generator.

The test consists of ten consecutive samples of 1000 numbers each for each of three different correlation coefficients.

THEORETICAL					MAXIMUM LIKELIHOOD ESTIMATES					
COVARIANCES		CORR. COEFF.	MEANS		COVARIANCES		CORR. COEFF.	MEANS		H
1.0	1.0	1.0	1.0	1.0	.9820	.9820	1.0000	.9672	.9672	
1.0	1.0	0.75	0.0	0.0	.9311	.9914	.7313	-.0093	.0047	.20
					1.0318	1.0417	.7731	-.0562	-.0464	1.56
					.9039	.8701	.7347	-.0407	-.0565	1.60
					.8551	.8862	.7340	-.0689	-.0866	3.76
					1.0332	1.0215	.7486	-.0252	-.0313	.49
					1.0300	1.0098	.7535	-.0861	-.1265	8.09
					.9627	1.0034	.7350	-.0538	-.0153	.17
					.9655	1.1056	.7606	-.0003	-.0021	.00
					1.0732	1.0957	.7639	-.0523	-.0348	1.39
					1.0387	1.0587	.7550	.0430	.0366	.95
1.0	1.0	0.50	0.0	0.0	.9311	1.0290	.4813	-.0093	.0107	.20
					1.0318	1.0138	.5398	-.0562	-.0337	1.60
					.9039	.8793	.4682	-.0407	-.0543	1.60
					.8651	.9090	.4787	-.0689	-.0802	3.76
					1.0332	1.0206	.4956	-.0252	-.0288	.49
					1.0300	1.0005	.5021	-.0861	-.1241	8.09
					.9627	1.0320	.4834	-.0538	.0059	2.17
					.9655	1.1277	.5401	-.0003	-.0026	.00
					1.0732	1.0822	.5265	-.0523	-.0204	1.39
					1.0387	1.0570	.5115	.0430	.0272	.95

THEORETICAL					MAXIMUM LIKELIHOOD ESTIMATES					
COVARIANCES		CORR. COEFF.	MEANS		COVARIANCES		CORR. COEFF.	MEANS		H
1.0	1.0	0.25	0.0	0.0	.9311	1.0497	.2417	-.0093	.0148	.20
					1.0318	.9844	.2961	-.0562	-.0030	1.60
					.9039	.8970	.2080	-.0407	-.0482	1.60
					.8652	.9255	.2316	-.0689	-.0683	3.76
					1.0332	1.0227	.2429	-.0252	-.0244	.49
					1.0300	.9967	.2483	-.0861	-.1122	8.09
					.9627	1.0490	.2400	-.0538	-.0232	2.17
					.9655	1.1124	.3202	-.0003	-.0028	.00
					1.0732	1.0631	.2834	-.0523	-.0067	1.39
					1.0387	1.0501	.2664	.0430	.0172	.95

Appendix VII

The FORTRAN programs to produce multivariate normal random vectors. The first entry must be 'CALL TRIANG!!'. This produces the triangularized matrix, and allows repetitive calls to 'CALL MULTN(Z)'. The argument 'Z' is the starting address of the random vector. Several other entries are necessary. The dimension of the covariance matrix (and the desired vectors) is set equal to 'NR'. The desired covariance matrix is stored as a matrix called 'C'. A common statement with 'NR' and 'C', with 'C' appropriately dimensioned, is included. 'Z' is also dimensioned. The following sample program will read in a matrix 'C', which is a 5 by 5 matrix punched column by column on cards. The random vectors are stored in a 5 by 100 array. Subroutine 'MULTN' and subroutine 'TRIANG' follow. Function 'ZMARS', the normal random number generator, from Appendix III must be added. Subroutine TRIANG makes external calls to SQRTF.

PROGRAM EXAMPLE

```
COMMON/NR/NR/C/C(5,5)/P/P(5,5)
DIMENSION Z(5), ARRAY(100,5)
NR=5
READ 101, ((C(I,J),I=1,NR),J=1,NR)
101 FORMAT (5F8.4)
CALL TRIANG
DO 111 J=1,100
CALL MULTN(Z)
DO 111 K=1,5
111 ARRAY(J,K)=Z(K)
STOP
END
```

For a three by three matrix subroutine TRIANG takes 10600 microseconds, and each vector is produced in 9520 microseconds.

It is noted that since the programs are written in FORTRAN,
they are not in any sense optimal.

```
      SUBROUTINE TRIANG
      COMMON/NR/NR/C/C(3,3)/P/P(3,3)
      NC=1
      DQ 5 I = 1, NR
      DQ 5 J = 1, NR
5     P(I,J) = 0.0
      P(1,1)=SQRTF(C(1,1))
      IF(P(1,1).LE..0001)771,9
9     DO 15 K=2,NR
15    P(K,1)=C(K,1)/P(1,1)
18    DO 83 JH=2,NR
      Q=0.0
      NC=NC+1
      NCM=NC-1
      NCP=NC+1
      DO 50 J=1,NCM
50     Q=Q+P(NC,J)*P(NC,J)
      X=C(JH,JH)-Q
      IF (X.LE..000005) 777,505
505    P(JH,JH)=SQRTF(X)
51     DO 55 K=NCP,NR
      S=0.0
      DO 53 JQ=1,NCM
53     S=S+P(K,JQ)*P(NC,JQ)
55     P(K,NC)=(C(K,NC)-S)/P(NC,NC)
83     CONTINUE
      GO TO 666
771   DO 773 L=1,NR
773   P(L,1)=0.0
      GO TO 18
777   DO 780 L=JH,NR
780   P(L,JH)=0.0
      GO TO 83
666   CONTINUE
      RETURN
      END
```



```

SUBROUTINE MULTN(Z)
COMMON/NR/NR/P/P(3,3)
DIMENSION Z(10),RZM(10)
DO 220 J=1, NR
  Z(J)=0.0
220 RZM(J)=ZMARS(DUM)
  DO 230 L=1,NR
    DO 230 M=1,NR
230 Z(L)=Z(L)+P(L,M)*RZM(M)
  RETURN
END

```


Appendix VIII

The tests are for a sample size of 100 vectors.

The first matrix tested was a 10 by 10 identity matrix. The maximum likelihood estimate of the covariance matrix was:

.814	-.048	.049	.127	.085	.001	.073	-.110	-.129	-.081
-.048	1.112	.133	.079	.245	-.040	.152	.017	-.065	-.011
.049	.133	.836	.133	-.017	.035	.071	-.023	-.131	-.185
.127	.079	.133	1.032	.074	-.027	.114	-.067	-.090	-.270
.085	.245	-.017	.074	.972	-.030	-.034	-.036	.108	.096
.001	-.040	.035	-.027	-.030	1.059	-.021	-.034	-.085	-.025
.073	.152	.071	.114	-.034	-.021	.970	.001	-.136	-.077
-.110	.017	-.023	-.067	-.036	-.034	.001	.966	.147	.045
-.129	-.065	-.131	-.090	.108	-.085	-.136	.147	1.047	-.009
-.081	-.011	-.185	-.270	.096	-.025	-.077	.045	-.009	1.075

Covariance Matrix Input	Triangularized Matrix	Covariance Matrix Estimate
$\begin{bmatrix} 2 & 1 & 2 \\ 1 & 3 & 3 \\ 2 & 3 & 4 \end{bmatrix}$	$\begin{bmatrix} 1.414 & 0 & 0 \\ .707 & 1.581 & 0 \\ 1.414 & 1.265 & .632 \end{bmatrix}$	$\begin{bmatrix} 2.227 & 1.154 & 2.318 \\ 1.154 & 3.006 & 3.066 \\ 2.318 & 3.066 & 4.245 \end{bmatrix}$
$\begin{bmatrix} 2 & 1 & 2 & 3 \\ 1 & 3 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 7 \end{bmatrix}$	$\begin{bmatrix} 1.414 & 0 & 0 & 0 \\ .707 & 1.581 & 0 & 0 \\ 1.414 & 1.265 & .732 & 0 \\ 2.121 & 1.581 & .000 & 0 \end{bmatrix}$	$\begin{bmatrix} 1.69 & .79 & 1.62 & 2.48 \\ .79 & 3.28 & 3.16 & 4.07 \\ 1.62 & 3.16 & 3.91 & 4.78 \\ 2.48 & 4.07 & 4.78 & 6.55 \end{bmatrix}$
$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & -2 \\ 0 & -1 & 3 & 1 \\ 0 & -2 & 1 & 4 \end{bmatrix}$	$\begin{bmatrix} 1.000 & 0 & 0 & 0 \\ 0 & 1.414 & 0 & 0 \\ 0 & -.707 & 1.581 & 0 \\ 0 & -1.414 & .000 & 1.414 \end{bmatrix}$	$\begin{bmatrix} .848 & -.038 & -.031 & .064 \\ -.038 & 2.335 & -.998 & -2.673 \\ -.031 & -.998 & 2.769 & 1.138 \\ .064 & -2.673 & 1.138 & 5.169 \end{bmatrix}$
$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & -2 \\ 0 & -1 & 3 & 1 \\ 0 & -2 & 1 & 4 \end{bmatrix}$	$\begin{bmatrix} 1.000 & 0 & 0 & 0 \\ 0 & 1.414 & 0 & 0 \\ 0 & -.707 & 1.581 & 0 \\ 0 & -1.414 & .000 & 1.414 \end{bmatrix}$	$\begin{bmatrix} .950 & .014 & -.272 & -.008 \\ .014 & 2.202 & -.775 & -2.155 \\ -.272 & -.775 & 2.501 & .614 \\ -.008 & -2.155 & .614 & 3.972 \end{bmatrix}$

Covariance
Matrix Input

$$\begin{bmatrix} 5 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 4 \end{bmatrix}$$

Triangularized
Matrix

$$\begin{bmatrix} 2.236 & 0 & 0 \\ .447 & 1.673 & 0 \\ .894 & .359 & 1.752 \end{bmatrix}$$

Covariance Matrix
Estimate

$$\begin{bmatrix} 5.568 & 1.181 & 2.498 \\ 1.181 & 2.948 & .926 \\ 2.498 & .926 & 4.164 \end{bmatrix}$$

$$\begin{bmatrix} 4.282 & 1.221 & 1.512 \\ 1.221 & 2.882 & 1.119 \\ 1.512 & 1.119 & 4.612 \end{bmatrix}$$

$$\begin{bmatrix} 3.735 & .729 & 1.694 \\ .729 & 3.070 & 1.471 \\ 1.694 & 1.471 & 3.680 \end{bmatrix}$$

$$\begin{bmatrix} 7.481 & 1.169 & 2.773 \\ 1.169 & 3.172 & .656 \\ 2.773 & .656 & 4.046 \end{bmatrix}$$

$$\begin{bmatrix} 4.663 & 1.006 & 1.777 \\ 1.006 & 3.034 & 1.212 \\ 1.777 & 1.212 & 3.647 \end{bmatrix}$$

Appendix IX.

The Poisson distributed random number generator.

The numbers are generated by an initial use of the variable 'NPOISSET(mean)', which initializes the generator for the desired value of the mean. This is followed by calls to 'NPOIS(DUM)' to actually produce the random numbers. The argument 'DUM' is not used. For example for a desired mean of 3.0:

```
X(1) = NPOISSET(3.0)
```

```
•
•
```

```
X(J) = NPOIS(DUM)
```

External calls are made to EXPF.

The observed average generation time per number was found to be approximately representable by: $TIME = 600 + 630(MEAN)$.

	IDENT	NPOIS
	ENTRY	NPOISSET,NPOIS
R	OCT	1777777777777777
UNIFORM	SLJ	**
	ENQ	0
	LDA	R
	LLS	19
	SCL	=040000000000000000
	ADD	R
	ADD	R
	ADD	R
	STA	R
	ARS	11
	ADD	=020000000000000000
	FAD	=020000000000000000
	SLJ	UNIFORM
NPOISSET	SLJ	**
	LDA	*
	ALS	+24
	SAU	**+2
+	INA	+1
	SAU	EXIT+1
+	LDA	**

	ALS	+24
	SAU	**+1
+	LDA	**
	SCM	=077777777777777777
	CALL	EXPF
+	STA	=ST1
	SLJ	START
NPOIS	SLJ	**
	LDA	*
	ALS	+24
	INA	+1
	SAU	EXIT+1
START	SIL	1 EXIT
	ENI	1 0
	RTJ	UNIFORM
+	THS	T1
	SLJ	A1
	SLJ	EXIT
A1	STA	=SY
	RTJ	UNIFORM
	INI	1 +1
	FMU	Y
	THS	T1
	SLJ	A1
EXIT	ENA	1 0
	ENI	1 **
	SLJ	**
	END	

Appendix X.

Test results for the Poisson generator.

The test consists of ten consecutive samples of either 1000 or 5000 numbers each-for various parameter values.

PARAMETER	M1	M2	M3	M4	d/n	CHIX	SAMPLE SIZE
0.5	.5010	.4965	.4713	1.1076			1000
	.5150	.5383	.5993	1.6819			
	.4790	.4720	.4909	1.2574			
	.5070	.5105	.5387	1.4413			
	.4950	.5125	.5571	1.4485			
	.4960	.4785	.4080	.8790			
	.4800	.4641	.4171	.9417			
	.4620	.4911	.5190	1.1739			
	.4860	.4683	.4465	1.0879			
	.5010	.5105	.5223	1.3104			
theor.	.5	.5	.5	1.25			
1.0	.9850	.9998	.9388	3.5138	.0131	2.1235	1000
	.9940	1.0690	1.1868	4.7322	.0181	4.7010	
	.9790	.9755	1.0516	4.2285	.0123	4.7794	
	1.0250	.9613	.8375	3.3099	.0209	4.9883	
	.9990	.9880	.9408	3.7028	.0058	2.1281	
	.9610	.9004	.7044	2.7304	.0193	9.8119	
	1.0030	.9539	.8249	3.2231	.0079	2.3483	
	.9740	.9523	.8548	3.5808	.0161	14.0873	
	.9900	.9769	.9100	3.3157	.0152	7.1024	
	1.0310	1.0571	1.1061	4.4782	.0117	2.3473	
theor.	1.0	1.0	1.0	4.0	.0386(10%)	11.651(10%)	
2.0	1.9590	2.0994	2.5166	17.4173	.0193		1000
	1.9690	2.0120	2.3470	15.9226	.0143		
	2.0540	2.0031	1.6185	11.7484	.0200		
	1.9610	1.7893	1.2660	9.2416	.0109		
	2.0190	2.0687	2.2118	16.0789	.0130		
	1.9610	1.9374	2.1740	13.7535	.0220		
	1.9840	1.8996	1.8807	12.3351	.0213		
	1.9220	1.8618	1.5925	10.5854	.0290		
	2.0120	2.2381	2.6755	18.2820	.0147		
	1.9990	2.0110	2.0572	13.8852	.0073		
theor.	2.0	2.0	2.0	14.0	.0386(10%)		

PARAMETER	M1	M2	M3	M4	d/n	Sample Size	Generation Time (microsecs)
4.0	3.9578	3.8668	3.5334	47.2482	.013	5000	3128
	4.0214	4.1002	4.1851	54.9297	.008		
	4.0204	4.0684	3.6521	51.6736	.012		
	4.0422	4.0116	4.0824	51.6602	.010		
	3.9770	4.1349	4.6342	59.4956	.010		
	3.9880	4.0679	4.4847	58.0892	.006		
	3.9920	4.0664	4.7496	58.4535	.005		
	4.0130	3.8916	3.1439	46.6069	.014		
	4.0318	4.0536	4.1740	53.8986	.008		
	3.9806	3.9426	3.9144	50.7897	.005		
Theor.	4.0	4.0	4.0	52.0	.017 (10%)		
5.0	4.9340	4.9742	5.3430	78.8045	.016	1000	3735
	5.0462	5.0975	4.9371	80.2735	.014		
	5.0108	4.9345	4.6862	74.9158	.005		
	4.9912	5.0533	5.1257	80.9384	.004		
	5.0038	5.0412	5.7602	87.0035	.003		
	5.0126	5.0367	5.3394	83.8973	.005		
	5.0346	4.9844	4.8206	80.1724	.010		
	5.0016	5.0142	5.1778	77.7648	.006		
	5.0214	4.9419	4.7638	75.7139	.006		
	4.9598	5.0140	4.8610	77.9824	.009		
Theor.	5.0	5.0	5.0	80.0	.039 (10%)		
10.0	9.9760	9.6751	10.0922	276.3385	.013	1000	6935
	9.8280	9.5560	9.8887	280.6076	.030		
	9.9070	10.4328	12.0257	322.7622	.027		
	10.1540	9.8441	9.4111	287.9662	.019		
	10.1090	10.8179	10.5374	347.2039	.021		
	9.9710	9.9922	8.4548	305.9784	.012		
	10.0520	10.2676	11.4099	298.6314	.029		
	10.1500	10.3258	11.2464	307.3533	.020		
	9.9940	9.4594	7.1034	265.9739	.017		
	10.0340	9.6445	8.7856	283.3459	.011		
Theor.	10.0	10.0	10.0	310.0	.39 (10%)		

PARAMETER	M1	M2	M3	M4	d/n	Sample Size
5.0	4.9340	4.9742	5.3430	78.8045	.016	5000
	5.0462	5.0975	4.9371	80.2735	.014	
	5.0108	4.9345	4.6862	74.9158	.005	
	4.9912	5.0533	5.1257	80.9384	.004	
	5.0038	5.0412	5.7602	87.0035	.003	
	5.0126	5.0367	5.3394	83.8973	.005	
	5.0346	4.9844	4.8206	80.1724	.010	
	5.0016	5.0142	5.1778	77.7648	.006	
	5.0214	4.9419	4.7638	75.7139	.006	
	4.9598	5.0140	4.8610	77.9824	.009	
Theor.	5.0	5.0	5.0	80.0	.017 (10%)	
	9.9948	10.0720	10.4326	303.7092	.006	5000
	10.0402	9.9338	9.4176	292.8100	.011	
	10.0174	10.1211	12.3246	335.7205	.004	
	10.0260	9.8721	9.3898	296.6628	.016	
	10.0852	9.8547	9.8097	302.7969	.017	
	10.0176	9.8136	9.8867	294.7293	.011	
	9.9914	10.1918	11.8066	323.7264	.006	
Theor.	10.0	10.0	10.0	310.0	.017 (10%)	

Appendix XI.

Exponential random number generator.

The numbers are generated by a call to 'EXPRN(DUM)'. The argument

'DUM' is not used. For example: $Y = \text{EXPRN}(\text{DUM}) + 4.2$

No external calls are made. The observed average generation time

for one number is 2270 microseconds. A conversion for numbers

with parameter other than one is given on page 31 in Section 8.1.

	IDENT	EXPRN
	ENTRY	EXPRN
P	DEC	1.00000000
	DEC	.99999984
	DEC	.99999824
	DEC	.99998381
	DEC	.99986834
	DEC	.99906004
	DEC	.99421023
	DEC	.96996120
	DEC	.87296508
	DEC	.58197672
Q	DEC	1.00000000
	DEC	.99999989
	DEC	.99999970
	DEC	.99999917
	DEC	.99999774
	DEC	.99999386
	DEC	.99998330
	DEC	.99995460
	DEC	.99987659
	DEC	.99966454
	DEC	.99908812
	DEC	.99752125
	DEC	.99326205
	DEC	.98168436
	DEC	.95021293
	DEC	.86466471
	DEC	.63212055
TABLE	BSS	20
MIN	DEC	0.0
R	OCT	5402033450727422
UNIFORM	SLJ	**
	ENQ	0
	LDA	R
	LLS	19

	SCL	=0400000000000000
	ADD	R
	ADD	R
	ADD	R
	STA	R
	ARS	11
	ADD	=0200000000000000
	FAD	=0200000000000000
	SLJ	UNIFORM
EXPRN	SLJ	**
	LDA	*
	ALS	24
	INA	1
	SAL	A4+1
	SIL	1 A4
	SIU	2 A4+1
	ENI	1 +10
	RTJ	UNIFORM
+	THS	1 P
	SLJ	A4
+	ENI	2 -1
A1	INI	2 +1
	RTJ	UNIFORM
+	STA	2 TABLE
+	ISK	1 +9
	SLJ	A1
+	STA	MIN
A2	IJP	2 A31
	SLJ	A3
A31	LDA	2 TABLE
	THS	MIN
	SLJ	A2
	STA	MIN
	SLJ	A2
A3	LDA	MIN
+	ENI	1 +17
	RTJ	UNIFORM
+	THS	1 Q
	SLJ	A4
	ENA	1 0
	INA	-16
	SCM	=0777777777777777
	ADD	=0204400000000000
	FAD	=0204400000000000
A4	FAD	MIN
	ENI	1 **
	ENI	2 **
	SLJ	**
	END	

Appendix XII.

Test results for the exponential generator.

The test consists of consecutive samples of 1000 each for lambda equal one.

SAMPLE	M1	M2	M3	M4	d/n	Generation Time
1000-1	1.0056	.9544	1.5731	6.0113	.015	
-2	1.0602	1.1981	2.9340	15.0479	.020	
-3	1.0606	1.2050	2.9526	15.5327	.037	
-4	1.0272	1.1211	2.5961	12.1087	.020	
-5	.9422	.8821	1.7657	7.6101	.030	
-6	1.0244	.9484	1.4382	5.2094	.031	
-7	.9273	.8457	1.4637	5.7615	.038	
-8	1.0520	1.1072	2.4988	12.2010	.033	
-9	1.0323	1.2839	3.8978	23.6524	.016	
-10	.9751	.8368	1.3691	5.3359	.022	2270 microsecs
Theor.	1.0	1.0	2.0	9.0	.039 (10%)	
-11	.9854	1.0562	2.4394	11.6124	.018	
-12	.9998	1.0495	2.2430	10.3795	.014	
-13	.9841	1.0447	2.2683	10.1315	.037	
-14	1.0204	1.0517	1.9077	7.1153	.021	
-15	.9513	.7981	1.1864	4.1650	.022	
-16	.9592	.9722	1.7937	6.9373	.041	
-17	1.0403	1.1069	2.2454	9.9354	.017	
-18	1.0329	1.0635	1.9497	7.6527	.018	
-19	.9711	.9014	1.6697	6.8324	.022	
-20	.9835	.9451	1.7445	7.1993	.020	
-21	1.0051	1.0281	2.1547	9.4030	.015	
-22	.9631	.8759	1.3956	4.9154	.020	
-23	.9951	.9813	1.7784	7.1579	.015	
-24	1.0176	1.1237	2.5168	12.1998	.020	
-25	1.0278	1.0664	2.2520	10.2068	.025	
-26	1.0317	1.0396	2.1303	9.3136	.027	
-27	1.0188	.9595	1.6659	6.7573	.024	
-28	.9334	.8338	1.3261	4.7659	.038	
-29	.9468	.9008	1.8411	8.4889	.034	
-30	1.0172	.9759	1.8041	7.5746	.029	

Addendum 1.

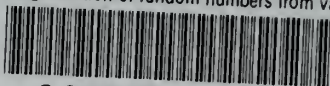
Results of the investigation of the tails of the Marsaglia normal random number generator.

The test results belie the idea resulting from investigations made in Section 4.3. A typical series of results of the tests are shown below. The first line gives the theoretical cumulative sample result based on a sample size of 100. The following data shows the experimental results. The first interval is from minus infinity to -2.56, following intervals are 0.16 in width. Thus only the negative half of the distribution is shown.

.52	.82	1.25	1.88	2.74	3.92	5.48	7.49	10.03	13.14	16.85	21.19	26.11
1	1	3	3	3	5	7	8	10	11	15	17	27
0	0	0	0	1	1	1	1	1	5	11	15	18
1	1	1	1	2	2	3	4	8	12	17	20	23
0	1	1	4	6	10	12	13	15	16	19	26	33
0	1	2	4	5	6	8	10	12	16	22	26	29
1	1	1	3	4	7	9	10	13	14	16	21	28
0	0	0	1	1	1	3	7	11	13	18	21	25
0	0	1	2	3	5	6	8	10	13	17	19	25
0	0	3	3	4	6	9	12	12	14	18	26	30
1	1	3	5	7	8	8	8	8	11	17	21	26

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The generation of random numbers from va



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